1. Interpolating polynomials

Definitions: (interval, continuous function, abscissas, and polynomial)

$x \in \mathbf{R}$, real number; $I = [a, b] \subset \mathbf{R}$, an interval;
$f : I \to \mathbf{R}$, continuous function.

$x_0, x_1, x_2, \cdots, x_n \in I$ n+1 distinct points (abscissas).

Polynomial of degree n, $p_n(x)$, is a linear combination of $\{1, x, x^2, \cdots, x^n\}$

$p_n(x) := a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

Theorem. (existence and uniqueness of interpolating polynomial)
There exists a unique polynomial of degree at most n, $p_n(x)$, that satisfies

$$p_n(x_i) = f(x_i), \quad \text{for each } i = 0, \cdots, n.$$

We call $p_n(x)$ the interpolating polynomial.

Exc1-1) Prove the above theorem.

• Lagrange form of interpolating polynomial.
(Has a simple form and useful for the error estimation.)

Derive an interpolating polynomial for points, $(x_i, f_i),\ i = 0, \cdots n, \quad f_i := f(x_i)$

Defining the Lagrange polynomial by

$$L_{n,i}(x) := \frac{(x - x_0)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)} = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}$$

Lagrange form of interpolating polynomial is written

$$p_n(x) = \sum_{i=0}^{n} L_{n,i}(x)\, f_i$$

Theorem: (Interpolation Error)
If a function f is continuous on [a,b] and has n+1 continuous derivatives on (a,b), then for ∀ x∈[a,b], ∃ ξ(x)∈(a,b), such that

$$f(x) = p_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x - x_i)$$

• Newton form of interpolating polynomial.

$$\begin{aligned} p_{0,\cdots,n}(x) &= a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots \\ &\quad + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}) \\ &= a_0 + \sum_{k=1}^{n} a_k \prod_{i=0}^{k-1}(x-x_i) \end{aligned}$$

We construct an interpolating polynomial for f(x) in the above form, that is,
$p_{0,\cdots,n}(x)$ satisfies $p_{0,\cdots,n}(x_k) = f(x_k)$.

Definition (Divided difference)

The zeroth divided difference w.r.t. the point x_i is written $f[x_i] = f(x_i)$.

The kth divided difference of f w.r.t. the points $x_i, x_{i+1}, \cdots, x_{i+k},$ is

$$f[x_i, x_{i+1}, \cdots, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \cdots, x_{i+k}] - f[x_i, x_{i+1}, \cdots, x_{i+k-1}]}{x_{i+k} - x_i}$$

$$0 < k \leq n,$$

• Newton form of interpolating polynomial is written

$$p_{0,\cdots,n}(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \cdots, x_k] \prod_{i=0}^{k-1} (x - x_i).$$

namely, $a_k = f[x_0, x_1, \cdots, x_k]$

Newton form is more efficient; fewer operation to determine its coefficients. Particularly, when a new data points become available, Newton form allows them to be incorporated easily.

• Interpolation error in Newton form can be derived as follows:

If $f(x) \in C^{n+1}(a,b)$, $\exists p_{0,\cdots,n+1}(x)$ with abscissas $x_0, \cdots, x_n, t$

$$p_{0,\cdots,n+1}(x) = p_{0,\cdots,n}(x) + f[x_0, x_1, \cdots, x_n, t] \prod_{i=0}^{n} (x - x_i).$$

At the point t, $f(t) = p_{0,\cdots,n+1}(t)$. Writing $t = x$,

$$f(x) - p_{0,\cdots,n}(x) = f[x_0, \cdots, x_n, x] \prod_{i=0}^{n} (x - x_i).$$

Exc 1-2) Derive the Newton form of interpolating polynomial,

$$p_{0,\cdots,n}(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \cdots, x_k] \prod_{i=0}^{k-1} (x - x_i).$$

Exc 1-3) Show, for any permutation $m_0, m_1, \cdots, m_k$ of $0, 1, \cdots, k$

$$f[x_{m_0}, x_{m_1}, \cdots, x_{m_k}] = f[x_0, x_1, \cdots, x_k]$$

Exc 1-4) Check that the interpolating error formula in Newton form

$$f(x) = p_{0,\cdots,n}(x) + f[x_0, \cdots, x_n, x] \prod_{i=0}^{n} (x - x_i).$$

is identical to

$$f(x) = p_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

hint: apply the generalized Rolle's theorem to $g(x) := f(x) - p_{0,\cdots,n}(x)$

to show $\exists \xi \in (a,b)$, s.t. $f[x_0, \cdots, x_n] = \dfrac{f^{(n)}(\xi)}{n!}$, Limitation of the interpolating polynomials

- Runge's phenomenon.

  When approximating the function f(x) on [a,b] by an interpolating polynomial, an error does not necessary decrease as increase the degree of polynomial. The interpolation oscillates to the end of the interval,

$$f(x) = \frac{1}{1+25x^2}, \text{ defined on an interval } [-1,1],$$

$$\lim_{n\to\infty}\left(\max_{-1\le x\le 1}|f(x)-P_n(x)|\right)=\infty.$$

Also consider a function $f(x)=\sqrt{|x|}$, on $[-1,1]$,

which is singular at x = 0.

cf) Gibbs phenomenon

When approximating a periodic piecewise differentiable function f(x) by the Fourier series, an error near to the discontinuity of f(x) does not decrease as increasing the number of Fourier series.

Some more theorems.

Theorem: (Weierstrass)

$\forall f(x) \in C[a,b]$, and $\forall \varepsilon > 0$,
$\exists p(x)$ a polynomial such that $|p(x) - f(x)| < \varepsilon$ for $\forall x \in [a,b]$.

Idea of a proof) A following polynomial has this property.

$$p_n(x) := \sum_{i=0}^{n} f(i/n) b_{n,i} = \sum_{i=0}^{n} f(i/n)\, {}_nC_i\, x^i (1-x)^{n-i},$$

The Bernstein polynomials $\{b_{n,i}\}$ converges uniformly to f(x) on [0,1]

Theorem: (Faber)

There is no universal node matrix (which is a sequence of abscissas with increasing points), for which the corresponding interpolation polynomials converges to $\forall$ f(x)$\in$C[a,b] .

How to overcome the problem.

(1) Use optimal points for abscissas for the interpolation:

Chebyshev points (roots of Chebyshev polynomial) minimize ℓ_∞ norm,

$$||f(x) - p_n(x)||_\infty := \max_{x\in[a,b]} |f(x) - p_n(x)|$$

Roots of Legendre polynomial minimize ℓ_2 norm,

$$||f(x) - p_n(x)||_2 := \left[\int_a^b [f(x) - p_n(x)]^2 dx\right]^{1/2}$$

(2) Use piecewise polynomial interpolation with lower degree, such as

Piecewise linear interpolation, Spline interpolation,

Hermite interpolation.

ex) Cubic Hermite: Interpolation $s_i(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

Interpolate $f(x)$ on $[a, b]$ with subintervals $[x_i, x_{i+1}]$ assuming that $f(x)$ and $f'(x)$ are given at each x_i. Each subinterval $[x_i, x_{i+1}]$ has different coeffients a_i $i = 0, 1, 2, 3$, which are determined from 4 conditions $f(x_i) = s_i(x_i)$, $f'(x_i) = s_i'(x_i)$ at $x = x_i$, and $f(x_{i+1}) = s_i(x_{i+1})$, $f'(x_{i+1}) = s_i'(x_{i+1})$ at $x = x_{i+1}$.

Exc1-5) Programing:

a). Make a code for the interpolation polynomial in Lagrange form and Newton form. (It is allowed to use a code from the lecture.)

b). Compare execution time. Check if your procedure is optimal.

c). Using Chebyshev points, estimate errors in ℓ_∞ and ℓ_2 norm, for different degrees of interpolating polynomial n such as $n = 2^n$, n =2 to 7

d). (optional) Using the roots of Legendre polynomial, redo c).

Exc1-6) Numerically confirm that the interpolating polynomial based on the Bernstein polynomial converges to the Runge's function on [-1,1].
(Note: the Bernstein polynomials in this note is defined on [0,1].)