A note on an initial data code for compact stars in numerical relativity.

Kōji Uryū (UWM)

This note :

Mathematical modeling of the quasiequilibrium solution for the rotating compact star, BNS, BH-NS and so on that may be used as the initial data.

Astrophysical spects is not included in this note.

- A review.
- Formulations of the problem.
- A numerical method.

Review : (quasi)equilibrium solutions

- A classifying catalogue for (quasi)equilibrium solutions.

Configuration
- Ellipsoidal figures
  - ··· (exact solution, virial method, variational method)
- Perturbative expansion ··············· (semi-)analytic
- Hydrostatic (stationary) equilibrium ··· numerical

EOS for fluid
- Incompressible fluid ··· $\rho = \text{const.}$
- Polytropic EOS ······· $p = \kappa\rho^{1+1/n}$
- nuclear EOS ·········· tabulated or fitting formula

Flow field
- Co-rotation (synchronous rotation)
- Irrotation (counter rotation)
- Arbitrary spin
- Differential rotation

Gravity
- Newtonian
- Post-Newtonian
- IWM formalism
- Stationary, axi-symmetric
- Waveless formalism
- Helical symmetry

Components
- A single star
- Equal mass
- Unequal mass
- BH-NS

3+1 decomposition and York-Lichnerowicz conformal decomposition.

$(\mathcal{M}, g_{\alpha\beta})$, globally hyperboric spacetimes that have a timelike t^α.

$\Sigma = \Sigma_0$: a Cauchy surface transverse to t^α.

$\gamma_{ab}(t)$: the spatial metric on Σ_t.

n^α: the future-pointing unit normal to this foliation.

Spacetime tensors that are orthogonal on all of their indices to n^α are identified with spatial tensors on Σ_t. In particular the projection of $\gamma_{\alpha\beta} = \gamma_{ab} + n_\alpha n_\beta$ orthogonal to n^α is associated with $\gamma_{ab}(t)$.

Since t^α is transverse to Σ, one can introduce non-vanisheng lapse α and a shift β^α, as $t^\alpha = \alpha n^\alpha + \beta^\alpha$, where $\beta^\alpha n_\alpha = 0$ a vector on Σ.

In a chart $\{t, x^i\}$, for which Σ is a $t =$ constant surface, the metric is

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt).$$

$g_{\alpha\beta}$ is 4D metric of $\mathcal{M}$, γ_{ab} is 3D spatial metric of Σ.

◦ A summary of notations.

t : parametrize foliation Σ_t.

t^α : a generator of a map $\chi_t : \Sigma_{t_0} \to \Sigma_t$, normalized as $t^\alpha \nabla_\alpha t = 1$.

n^α : the future-pointing unit normal to the foliation. $n_\alpha = -\alpha \nabla_\alpha t$.

α : a lapse. β^α : a shift. t^α relates to n^α by $t^\alpha = \alpha n^\alpha + \beta^\alpha$, $\beta^\alpha n_\alpha = 0$.

γ_{ab} : a spatial metric on Σ_t. (The projection tensor $\gamma_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta$.)

K_{ab} : the extrinsic curvature of Σ_t given by $K_{ab} = -\frac{1}{2}\pounds_n \gamma_{ab}$

A_{ab}: a trace free part of K_{ab}, K: a trace of K_{ab}, $A_{ab} := K_{ab} - \frac{1}{3}\gamma_{ab} K$.

$\tilde{\gamma}_{ab}$: the conformal metric defined by $\gamma_{ab} = \psi^4 \tilde{\gamma}_{ab}$.

$\tilde{A}_{ab}$: a conformal weighted A_{ab}, $A_{ab} := \psi^4 \tilde{A}_{ab}$, also $K_{ab} := \psi^4 \tilde{K}_{ab}$.

f_{ab} : the flat metric $h_{ab} = \tilde{\gamma}_{ab} - f_{ab}$:

D_a, $\tilde{D}_a$, $\overset{\circ}{D}_a$: the covariant derivatives associated with γ_{ab}, $\tilde{\gamma}_{ab}$ and f_{ab}.

○ Formulation.

∘ The field equation $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ is projected to Σ and its normal n^α.

Hamiltonian constraint : $(G_{\alpha\beta} - 8\pi T_{\alpha\beta}) n^\alpha n^\beta = 0,$
Momentum constraint : $(G_{\alpha\beta} - 8\pi T_{\alpha\beta}) \gamma^\alpha{}_a n^\beta = 0.$
Trace of a projection to Σ : $(G_{\alpha\beta} - 8\pi T_{\alpha\beta})(\gamma^{\alpha\beta} + n^\alpha n^\beta) = 0.$
Tr free part of a projection to Σ : $(G_{\alpha\beta} - 8\pi T_{\alpha\beta})(\gamma^\alpha{}_a \gamma^\beta{}_b - \frac{1}{3}\gamma_{ab}\gamma^{\alpha\beta}) = 0.$

Solved for the metric $\{\psi, \beta^a, \alpha, \tilde{\gamma}_{ab}\}$ on a slice Σ, in a chart $\{t, x^i\}$,

$$ds^2 = -\alpha^2 dt^2 + \psi^4 \tilde{\gamma}_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt).$$

∘ Gauge conditions:

(1) a temporal gauge (slicing condition, e.g. the maximal slice $K = 0$).
(2) three spatial gauge (e.g. the Dirac gauge $\overset{\circ}{D}_a \tilde{\gamma}^{ab} = 0$).

∘ A condition to specify the conformal decomposition:

$\det(\tilde{\gamma}_{ab}) = \det(f_{ab})$, ($f_{ab}$: a flat metric.)

($\tilde{\gamma}_{ab} := \psi^{-4}\gamma_{ab}$: the confomally rescaled spatial metric.)

∘ Evolution equations.

In 3+1 formulation used in numerical relativity, $\{\tilde{\gamma}_{ab}, \psi, \tilde{A}_{ab}, K\}$ are considered as "dynamical" variables. (cf. $\{\gamma_{ab}, \pi^{ab}\}$ in ADM.)

∘ From the definition, $K_{ab} = -\frac{1}{2}\pounds_n \gamma_{ab}$, with $\tilde{\gamma} = f$.

$$\begin{aligned} &\text{Trace part :} && (\partial_t - \pounds_\beta) \ln \psi^6 = -\alpha K, \\ &\text{Trace free part :} && (\partial_t - \pounds_\beta)\tilde{\gamma}_{ab} = -2\alpha \tilde{A}_{ab}. \end{aligned}$$

∘ Spatial components of the Einstein equation.

$(G_{\alpha\beta} - 8\pi T_{\alpha\beta})(\gamma^{\alpha\beta} + n^\alpha n^\beta) = 0$:

$$(\partial_t - \pounds_\beta)K = \alpha(\tilde{A}_{ab}\tilde{A}^{ab} - \frac{1}{3}K^2) - D_a D^a \alpha - 4\pi\alpha(\rho_{\mathrm{H}} + S),$$

$(G_{\alpha\beta} - 8\pi T_{\alpha\beta})(\gamma^\alpha{}_a \gamma^\beta{}_b - \frac{1}{3}\gamma_{ab}\gamma^{\alpha\beta}) = 0$:

$$(\partial_t - \pounds_\beta)\tilde{A}_{ab} = \alpha(K\tilde{A}_{ab} - 2\tilde{A}_{ac}\tilde{A}_b{}^c) + \psi^{-4}[\,\alpha R_{ab} - D_a D_b \alpha - 8\pi\alpha S_{ab}\,]^{\mathrm{TF}}$$

Note: some of $\pounds_\beta$ are operating to tensor densities with different weights.

Separate out a flat Laplacian for h_{ij} from R_{ij}.

Split R_{ab} as $R_{ab} = {}^3\tilde{R}_{ab} + {}^3\tilde{R}^{\psi}_{ab}$. ${}^3\tilde{R}_{ab}$: Ricci tensor associated with $\tilde{\gamma}_{ab}$.

$$ {}^3\tilde{R}^{\psi}_{ab} = -\frac{2}{\psi}\tilde{D}_a\tilde{D}_b\psi - \tilde{\gamma}_{ab}\frac{2}{\psi}\tilde{D}^c\tilde{D}_c\psi + \frac{6}{\psi^2}\tilde{D}_a\psi\tilde{D}_b\psi - \tilde{\gamma}_{ab}\frac{2}{\psi^2}\tilde{D}_c\psi\tilde{D}^c\psi\,. $$

$$ {}^3\tilde{R}_{ab} = -\frac{1}{2}\overset{\circ}{\Delta}h_{ab} - \frac{1}{2}[\overset{\circ}{D}_b(\tilde{\gamma}_{ac}F^c) + \overset{\circ}{D}_a(\tilde{\gamma}_{bc}F^c)] + \tilde{R}^{\mathrm{NL}}_{ab}. $$

$$ F^a \equiv \overset{\circ}{D}_b\tilde{\gamma}^{ab} = \overset{\circ}{D}_b h^{ab}, \quad \overset{\circ}{\Delta} = f^{ab}\overset{\circ}{D}_a\overset{\circ}{D}_b, \quad \tilde{\gamma}_{ab} = f_{ab} + h_{ab}, \quad \tilde{\gamma}^{ab} = f^{ab} + h^{ab}. $$

$$ \begin{aligned} R^{\mathrm{NL}}_{ab} \;=\; & -\frac{1}{2}(\overset{\circ}{D}_b h^{cd}\overset{\circ}{D}_c h_{ad} + \overset{\circ}{D}_a h^{cd}\overset{\circ}{D}_c h_{bd} + h^{cd}\overset{\circ}{D}_c\overset{\circ}{D}_d h_{ab}) - \overset{\circ}{D}_a C^c_{cb} \\ & + C^c_{ab}C^d_{dc} - C^d_{ac}C^c_{bd} - \frac{1}{2}[\overset{\circ}{D}_b(h_{ac}F^c) + \overset{\circ}{D}_a(h_{bc}F^c)] + F^c C_{c,ab} \end{aligned} $$

$$ C^c_{ab} \equiv \frac{\tilde{\gamma}^{cd}}{2}(\overset{\circ}{D}_a h_{db} + \overset{\circ}{D}_b h_{ad} - \overset{\circ}{D}_d h_{ab}) \quad \text{and} \quad C_{c,ab} \equiv \tilde{\gamma}_{cd}C^d_{ab}. $$

Dirac gauge $F^a = 0$, $\det\tilde{\gamma}_{ab} = \det f_{ab}$, $C^b_{ba} = 0$

∘ Role of the Bianchi identity in 3+1 formulation.

In 3+1 formulation, the (contracted) Bianchi identity, $\nabla_\beta G^\beta{}_\alpha = 0$, implies that the constraints are automatically satisfied in $D(S)$ the domain of dependence of $S \in \Sigma$ if the constraints are satisfied on S initially, and the fields are evolved solving the spatial part of the Einstein equations in $D(S)$.

This can be seen as follows: As a consequence of the Bianchi identity, we have $\nabla_\beta \mathcal{E}^\beta{}_\alpha = 0$, where $\mathcal{E}_{\alpha\beta} := G_{\alpha\beta} - 8\pi T_{\alpha\beta} = 0$.

$C := (G_{\alpha\beta} - 8\pi T_{\alpha\beta}) n^\alpha n^\beta$ (Hamiltonian constraint: $\mathcal{H} = -\frac{1}{2\sqrt{\gamma}} C$.)

$C_a := (G_{\alpha\beta} - 8\pi T_{\alpha\beta}) \gamma_a{}^\alpha n^\beta$ (Momentum constraint: $\mathcal{C}_a = -\frac{1}{2\sqrt{\gamma}} C_a$.)

$\mathcal{E}_{ab} := (G_{\alpha\beta} - 8\pi T_{\alpha\beta}) \gamma_a{}^\alpha \gamma_b{}^\beta$

Projections of $\nabla_\beta \mathcal{E}^\beta{}_\alpha = 0$ to n^α and the hypersurface Σ become

$$n^\alpha \nabla_\beta \mathcal{E}^\beta{}_\alpha = -\pounds_n C + KC + \frac{1}{\alpha^2} D_a(\alpha^2 C^a) + K^{ab}\mathcal{E}_{ab} = 0,$$

$$\gamma_a{}^\alpha \nabla_\beta \mathcal{E}^\beta{}_\alpha = -\pounds_n C_a + KC_a + C D_a \ln\alpha + \frac{1}{\alpha} D_b(\alpha \mathcal{E}^b{}_a) = 0,$$

Hence, $C = 0$ and $C_a = 0$ on $S \in \Sigma$ and $\mathcal{E}_{ab} = 0$ in $D(S)$, then $\partial_t C = 0$ and $\partial_t C_a = 0$, meaning that constraints are always satisfied in $D(S)$.

◦ 3+1 decomposition of the steress-energy tensor.

The decomposition of the stress energy tensor $T_{\alpha\beta}$ are defined by

$$\rho_{\mathrm{H}} := T_{\alpha\beta} n^\alpha n^\beta, \qquad j_a := -T_{\alpha\beta}\gamma_a{}^\alpha n^\beta,$$
$$S_{ab} := T_{\alpha\beta}\gamma_a{}^\alpha \gamma_b{}^\beta, \qquad S := T_{\alpha\beta}\gamma^{\alpha\beta} = S_{ab}\gamma^{ab}.$$

ex) A perfect fluid stress-energy tensor

$$T_{\alpha\beta} = (\epsilon + p) u_\alpha u_\beta + p g_{\alpha\beta},$$

where u^α: 4 velocity, p: fluid's pressure, ϵ: energy density.
Writing $u^\alpha = u^t(t^\alpha + v^\alpha)$ with a spatial velocity $v^\alpha n_\alpha = 0$, we have

$$u^\alpha n_\alpha = -\alpha u^t, \qquad u^\alpha \gamma_{\alpha a} = u^t(\beta_a + v_a).$$

$$\rho_{\mathrm{H}} := h\rho(\alpha u^t)^2 - p,$$
$$j_a := h\rho\alpha(u^t)^2(\beta_a + v_a),$$
$$S_{ab} := h\rho(u^t)^2(\beta_a + v_a)(\beta_b + v_b) + p\gamma_{ab},$$
$$S := h\rho[(\alpha u^t)^2 - 1] + 3p.$$

Initial data on a conformal flat slice

Initial data construction

Data on the initial slice has to satisfy constraints $(G_{\alpha\beta} - 8\pi T_{\alpha\beta})n^\alpha = 0$.

For the most of black hole initial data, only these four constraint equations are solved.

○ Isenberg-Wilson-Mathews (IWM) formulation.

• 4 constraints and the spatial trace of Einstein equations are solved for spatially conformally flat metric on a maximally embedded slice Σ,

$$ds^2 = -\alpha^2 dt^2 + \psi^4 f_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),$$

f_{ij}: flat metric.

(5 components of the metric coefficients are solved. Stationary condition in rotating frame is assumed for the fluid equations of motion.)

• IWM formulation agrees with GR in a static and spherically symmetric spacetime, and with the first post-Newtonian approximation.

○ Initial data in IWM formulation.

• For the metric, $ds^2 = -\alpha^2 dt^2 + \psi^4 f_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$, four constraints and the spatial trace of the Einstein equation are solved.

$(G_{\alpha\beta} - 8\pi T_{\alpha\beta}) n^\alpha n^\beta = 0$:

$$\mathring{\Delta}\psi + \frac{\psi^5}{8}\left(A_{ab}A^{ab} + \frac{2}{3}K^2\right) + 2\pi\psi^5 \rho_{\mathrm{H}} = 0,$$

$(G_{\alpha\beta} - 8\pi T_{\alpha\beta}) \gamma_a{}^\alpha n^\beta = 0$:

$$\mathring{\Delta}\tilde{\beta}_a + \frac{1}{3}\mathring{D}_a \mathring{D}_b \tilde{\beta}^b + 2\alpha A_a{}^b \mathring{D}_b \ln\frac{\psi^6}{\alpha} - \frac{4}{3}\alpha \mathring{D}_a K - 16\pi\alpha j_a = 0,$$

$(G_{\alpha\beta} - 8\pi T_{\alpha\beta})(\gamma^{\alpha\beta} + \frac{1}{2} n^\alpha n^\beta) = 0$:

$$\mathring{\Delta}(\alpha\psi) + \psi^5 \pounds_{t-\beta} K - \alpha\psi^5\left(\frac{7}{8}A_{ab}A^{ab} + \frac{5}{12}K^2\right) - 2\pi\alpha\psi^5(\rho_{\mathrm{H}} + 2S) = 0.$$

• We choose $K = 0 = \partial_t K$ (maximal slicing). Because the spatial metric is conformally flat, A_{ab} does not involve a time derivative of the spatial metric. $A_{ab} = \frac{\psi^4}{2\alpha}\left(\pounds_\beta f_{ab} - \frac{1}{3} f_{ab} f^{cd} \pounds_\beta f_{cd}\right)$.

○ An equation set for a coding.

$$
\begin{aligned}
\overset{\circ}{\Delta}\psi &= -\frac{\psi^5}{8}\tilde{A}_{ab}\tilde{A}^{ab} - 2\pi\psi^5\rho_{\mathrm{H}} \\
\overset{\circ}{\Delta}\tilde{\beta}_a + \frac{1}{3}\overset{\circ}{D}_a\overset{\circ}{D}_b\tilde{\beta}^b &= -2\alpha\tilde{A}_a{}^b\overset{\circ}{D}_b \ln\frac{\psi^6}{\alpha} + 16\pi\alpha j_a \\
\overset{\circ}{\Delta}(\alpha\psi) &= \alpha\psi^5\frac{7}{8}\tilde{A}_{ab}\tilde{A}^{ab} + 2\pi\alpha\psi^5(\rho_{\mathrm{H}} + 2S)
\end{aligned}
$$

$$
\tilde{A}_{ab} = \frac{1}{2\alpha}\left(\pounds_\beta f_{ab} - \frac{1}{3}f_{ab}f^{cd}\pounds_\beta f_{cd}\right) = \frac{1}{2\alpha}\left(\overset{\circ}{D}_a\tilde{\beta}_b + \overset{\circ}{D}_b\tilde{\beta}_a - \frac{2}{3}f_{ab}\overset{\circ}{D}_c\tilde{\beta}^c\right)
$$

Indexes of conformal weighted quantities (quantities with tilde) are raise and lowered using the conformal metric $\tilde{\gamma}_{ab}$.
$\tilde{\gamma}_{ab} = f_{ab}$ for spatially conformal flat initial data.

$$
\begin{gathered}
\tilde{\beta}^a := \beta^a, \quad \tilde{\beta}_a := \tilde{\gamma}_{ab}\tilde{\beta}^b = \psi^{-4}\gamma_{ab}\beta^b = \psi^{-4}\beta_a, \\
\tilde{A}_a{}^b := A_a{}^b, \quad \tilde{A}_{ab} := \tilde{\gamma}_{bc}A_a{}^c, \quad \tilde{A}^{ab} := \tilde{\gamma}^{ac}A_c{}^b, \\
(\tilde{A}_{ab}\tilde{A}^{ab} = \tilde{A}_a{}^b\tilde{A}_b{}^a = A_a{}^bA_b{}^a = A_{ab}A^{ab}).
\end{gathered}
$$

◯ Shibata decomposition for the momentum constraint.

Often, the vector elliptic operator in the Momentum constraint is decomposed to improve the accuracy.

For $\overset{\circ}{\Delta}\tilde{\beta}_a + \frac{1}{3}\overset{\circ}{D}_a\overset{\circ}{D}_b\tilde{\beta}^b = \mathcal{S}_a$, introduce $\tilde{\beta}_a = B_a + \frac{1}{8}\overset{\circ}{D}_a(B - x^b B_b)$ where x^a are corrdinates that satisfy $\overset{\circ}{D}_a x^b = f_a{}^b$.

Substituting the decomposition to the momentum constraint, we have

$$\overset{\circ}{\Delta}\tilde{\beta}_a + \frac{1}{3}\overset{\circ}{D}_a\overset{\circ}{D}_b\tilde{\beta}^b = \overset{\circ}{\Delta}B_a + \frac{1}{6}\overset{\circ}{D}_a(\overset{\circ}{\Delta}B - x^b\overset{\circ}{\Delta}B_b) = \mathcal{S}_a.$$

So, we solve elliptic equations $\overset{\circ}{\Delta}B_a = \mathcal{S}_a$ and $\overset{\circ}{\Delta}B - x^b\overset{\circ}{\Delta}B_b = 0$ separatly. Substituting the formar to the latter we have,

$$\overset{\circ}{\Delta}B_a = \mathcal{S}_a := -2\alpha\tilde{A}_a{}^b\overset{\circ}{D}_b \ln\frac{\psi^6}{\alpha} + 16\pi\alpha j_a,$$

$$\overset{\circ}{\Delta}B = x^a\mathcal{S}_a.$$

Stationary condition for the fluid.

○ Formulation for the fluid.

A perfect fluid is described by its 4 velocity u^α and stress-energy tensor

$$T_{\alpha\beta} = (\epsilon + p)u_\alpha u_\beta + p g_{\alpha\beta},$$

where p is the fluid's pressure, ϵ its energy density.

As a consequence of the Bianchi identity, we have $\nabla_\beta T_\alpha{}^\beta = 0$.
A projection of the identity $\nabla_\beta T_\alpha{}^\beta = 0$ transverse to u^α with the projection tensor $q_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$, and its projection to u^α, give the relativistic Euler equation, and the mass energy conservation law.

Writing the identity $\nabla_\beta T_\alpha{}^\beta = 0$,

$$\nabla_\beta T_\alpha{}^\beta \;=\; (\epsilon + p)(u^\beta \nabla_\beta u_\alpha + u_\alpha \nabla_\beta u^\beta) + q_\alpha{}^\beta \nabla_\beta p + u_\alpha u^\beta \nabla_\beta \epsilon = 0,$$

these projections are derived

$q_\alpha{}^\gamma \nabla_\beta T_\gamma{}^\beta = 0$:

$$q_\alpha{}^\gamma [\,(\epsilon + p)u^\beta \nabla_\beta u_\gamma + \nabla_\gamma p\,] = 0 \quad \Leftrightarrow \quad q_\alpha{}^\beta [\,\pounds_u u_\beta + \frac{1}{\epsilon + p}\nabla_\beta p\,] = 0$$

$u^\alpha \nabla_\beta T_\alpha{}^\beta = 0$: $\quad u^\alpha \nabla_\alpha \epsilon + (\epsilon + p)\nabla_\alpha u^\alpha = 0 \;\;\Leftrightarrow\;\; \pounds_u \epsilon + (\epsilon + p)\pounds_u \sqrt{-g} = 0$

When the fluid is close to an equilibrium, one can derive a simpler set of equations. Introducing the baryon mass density ρ, and the specific enthalpy defined by $h := \dfrac{\epsilon + p}{\rho}$, the identity $\nabla_\beta T_\alpha{}^\beta = 0$ can be written

$$
\begin{aligned}
\nabla_\beta T_\alpha{}^\beta &= \rho[\, u^\beta \nabla_\beta (h u_\alpha) + \nabla_\alpha p \,] + h u_\alpha \nabla_\beta (\rho u^\beta) \\
&= \rho[\, u^\beta \nabla_\beta (h u_\alpha) + \nabla_\alpha h \,] + h u_\alpha \nabla_\beta (\rho u^\beta) - \rho\Big(\nabla_\alpha h - \frac{1}{\rho}\nabla_\alpha p\Big) \\
&= \rho[\, u^\beta \nabla_\beta (h u_\alpha) + \nabla_\alpha h \,] + h u_\alpha \nabla_\beta (\rho u^\beta) - \rho T \nabla_\alpha s,
\end{aligned}
$$

where s is the specific entropy. In the last line, the local 1st law of thermodynamics $dh = T ds + \dfrac{1}{\rho} dp$ is used; the last line is correct only for the reversible process. (This should be exact for the perfect fluid that has no entropy production).

Note that a projection $u^\alpha [\, u^\beta \nabla_\beta (h u_\alpha) + \nabla_\alpha h \,] = 0$ is trivial. Therefore, independent components of the equation $u^\beta \nabla_\beta (h u_\alpha) + \nabla_\alpha h = 0$ (which relates to the relativistic Euler eq.) are 3, not 4.

We assume that the baryon mass is conserved, $\nabla_\alpha(\rho u^\alpha) = 0$. Then, in the local thermodynamic equilibrium, a projection of $\nabla_\beta T_\alpha{}^\beta = 0$ to the 4 velocity u^α gives $u^\alpha \nabla_\alpha s = \pounds_u s = 0$.

Under these assumptions, the equations for the relativistic fluid become

$$
\begin{aligned}
& u^\beta \nabla_\beta (h u_\alpha) + \nabla_\alpha h = 0 && \Leftrightarrow && \pounds_u (h u_\alpha) + \nabla_\alpha h = 0 && \left(h := \frac{\epsilon + p}{\rho} \right) \\
& \nabla_\beta (\rho u^\beta) = 0 && \Leftrightarrow && \pounds_u (\rho \sqrt{-g}) = 0 && \\
& \nabla_\alpha s = 0 && \Leftrightarrow && \pounds_u s = 0 &&
\end{aligned}
$$

With an approprariate choice for EOS, a set of fluid equations is closed.

If the isentropic flow, $s =$ const everywhere in the fluid, is assumed, one can introduce the one-parameter EOS.

◯ Statinonary condition for the fluid.

⋆ We assume stationary state in the rotating frame for the fluid source. Impose a symmetry along $k^\alpha = t^\alpha + \Omega\phi^\alpha$ (Equilibrium assumption), with the $\Omega =$ constant.

$$\pounds_k(\rho u^t \sqrt{-g}) = 0, \quad \gamma_a{}^\alpha \pounds_k(h u_\alpha) = 0, \quad \text{or } \pounds_k(j_a \sqrt{\gamma}) = 0.$$

Intriducing the spatial velocity v^α, the 4 velocity is written

$$u^\alpha = u^t(k^\alpha + v^\alpha), \qquad v^\alpha n_\alpha = 0.$$

- Recall:

$$\begin{aligned}
t^\alpha &= \alpha n^\alpha + \beta^\alpha, \\
k^\alpha &= \alpha n^\alpha + \omega^\alpha = \alpha n^\alpha + \beta^\alpha + \Omega\phi^\alpha, \\
\omega^\alpha &= \beta^\alpha + \Omega\phi^\alpha, \quad \text{the shift in a rotating frame.}
\end{aligned}$$

∘ For corotational flow, $u^\alpha = u^t k^\alpha$, $v^\alpha = 0$, the rest mass conservation becomes trivial, and the relativistic Euler eq. has the first integral

$$\frac{h}{u^t} = \text{const.}$$

From the normalization of the four velocity $u_\alpha u^\alpha = -1$,

$$u^t = \frac{1}{\sqrt{\alpha^2 - \omega_a \omega^a}} = \frac{1}{\sqrt{\alpha^2 - \psi^4 f_{ab}\,\tilde{\omega}^a \tilde{\omega}^b}},$$

$$\text{where} \quad \tilde{\omega}^a = \tilde{\beta}^a + \Omega \phi^a.$$

cf) $u_\alpha u^\alpha = (u^t)^2 g_{\alpha\beta} k^\alpha k^\beta = (u^t)^2 g_{\alpha\beta}(\alpha n^\alpha + \omega^\alpha)(\alpha n^\beta + \omega^\beta) = (u^t)^2 g_{\alpha\beta}(-\alpha^2 + \omega_\alpha \omega^\alpha)$

(Exc. Consider how to formulate the differential rotation.)

∘ Source terms in the field equaitons for corotational flow.

Decomposition of the 4 velocity for the corotational flow $u^\alpha = u^t k^\alpha$ is

$$u^\alpha n_\alpha = -\alpha u^t$$
$$u^\alpha \gamma_{\alpha a} = u^t \omega_\alpha$$

$$\rho_{\mathrm{H}} := T_{\alpha\beta} n^\alpha n^\beta = h\rho(\alpha u^t)^2 - p,$$
$$j_a := -T_{\alpha\beta}\gamma_a{}^\alpha n^\beta = h\rho\alpha(u^t)^2\omega_a = h\rho\alpha(u^t)^2\psi^4\tilde{\omega}_a,$$
$$S_{ab} := T_{\alpha\beta}\gamma_a{}^\alpha\gamma_b{}^\beta = h\rho(u^t)^2\omega_a\omega_b + \gamma_{ab}\,p = h\rho(u^t)^2\psi^8\omega_a\omega_b + \psi^4\tilde{\gamma}_{ab}\,p,$$
$$S := T_{\alpha\beta}\gamma^{\alpha\beta} = h\rho[(\alpha u^t)^2 - 1] + 3\,p.$$

where $\quad \tilde{\omega}_a := \tilde{\gamma}_{ab}(\tilde{\beta}^b + \Omega\tilde{\phi}^b) = \tilde{\beta}_a + \Omega\tilde{\phi}_a$

∘ For irrotational flow, $hu_\alpha = \nabla_\alpha \Phi$,

$$D_a\left[\frac{\alpha\rho}{h}\left(D^a\Phi - hu^t\omega^a\right)\right] = 0,$$

$$v^a D_a\Phi + \frac{h}{u^t} = C = \text{const},$$

where $u^\alpha = u^t(k^\alpha + v^\alpha)$, $v^\alpha n_\alpha = 0$, $\omega^\alpha = \beta^\alpha + \Omega\phi^\alpha$.

For polytrope $p = \kappa\rho^{1+1/n}$, $h = 1 + (n+1)\dfrac{p}{\rho}$.

(u^t is solved from $u_\alpha u^\alpha = -1$.)

⋆ Velocity potential Φ is solved from the elliptic equation with the Neumann boundary condition.

$$v^a D_a h = 0, \text{ along the surface of NS.}$$

(the boundary condition is equivalent with$u^\alpha\nabla_\alpha h = 0$ with $\pounds_k h = 0$.)

Solving method for binary neutron stars :

A numerical method.

○ Simplistic chart for developing a numerical code.

Writing down all equations used in a numerical computation.
(Also important to look for the normalization of variables and the choice of parameters suitable for the numerical computation.)

$\downarrow$

Designing a numerical method
(e.g. the initial data code for the netron star:
Choice of coordinates, an elliptic solver,
and an iteration method, etc.)

$\downarrow$

Typing... perhaps about 3000 – 20000 lines by FORTRAN 77.

○ Normalization of the variables and a choice of parameters
— important for making a successful iteration scheme.

We have two parameters : $\{\Omega, C\}$. It is convenient to determine them by fixing two quantities; the separation and the central density.

one can additionally introduce a length scale R_0 for normalization. we take $2R_0$ to be the diameter of a NS. $\hat{r} := r/R_0$.

For a polytropic EOS, one can rescale (measure) the length scale by a constant κ ($p = \kappa\rho^{1+1/n}$) as $\bar{R}_0 := \kappa^{-n/2}R_0$.

Then all components of field equations are written as follows; for the fields ϕ ($= \{\psi, \alpha, \beta^a, h_{ab}\}$).

$$\overset{\circ}{\Delta}\phi = S_{\rm g}[\phi] + \bar{R}_0^2\, S_{\rm m}[\phi, \rho, \Phi],$$

where all quantities are normalized by R_0.
Fluid variable $\{\rho, \Phi\}$; Parameters $\{\Omega, C, \bar{R}_0\}$.

◯ Choice of coordinates and elliptic solver.

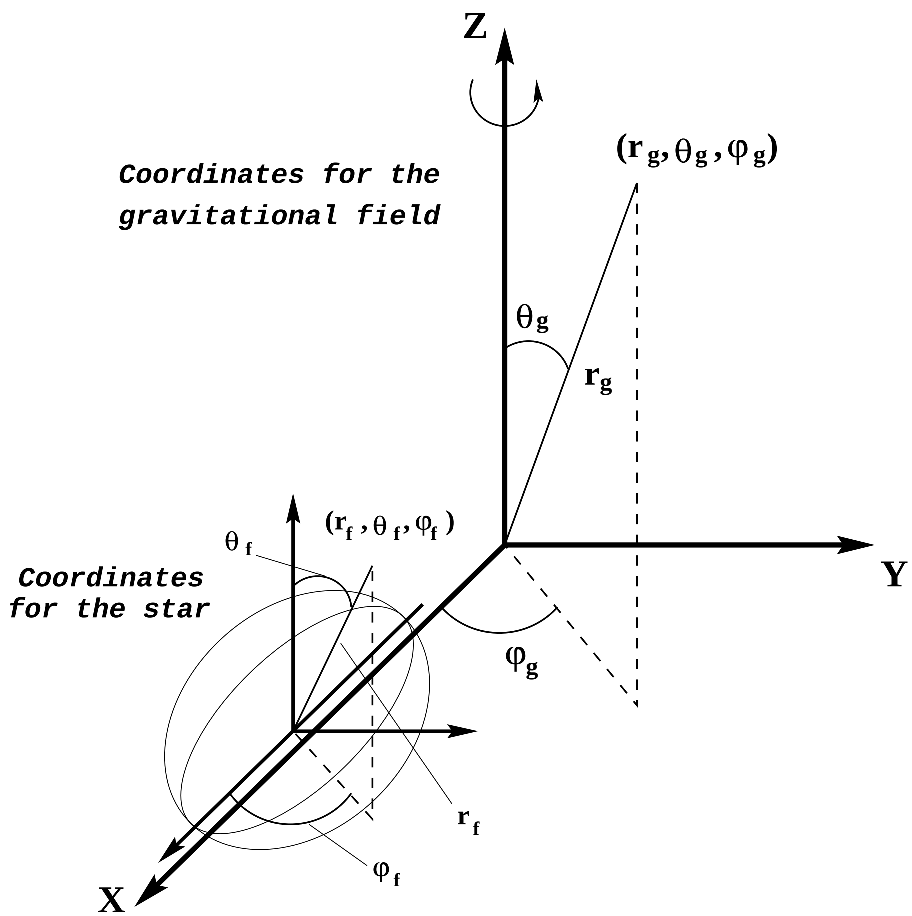
To solve the elliptic equations for the gravitational fields,

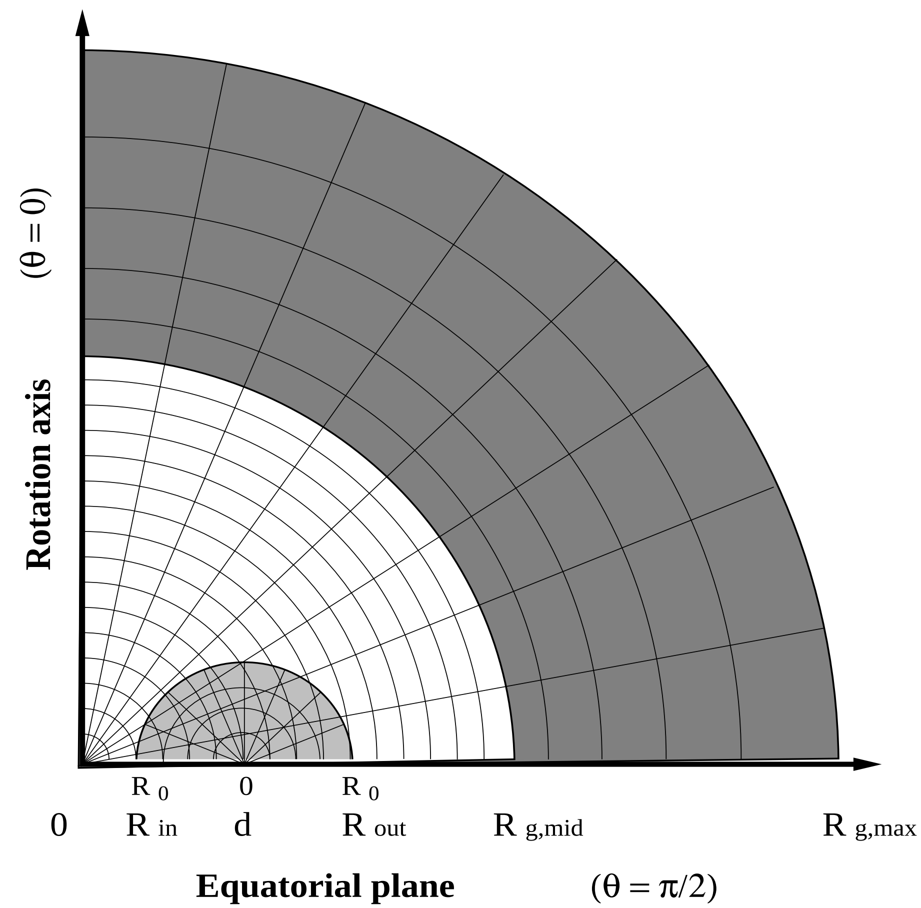
(1) any type of Poisson solver would work fine,
(2) coordinate choice may depend on a type of Poisson solver.

Our choice:
We choose spherical coordinates (r_g, θ_g, ϕ_g) whose origin is the center of orbital motion, and (r_f, θ_f, ϕ_f) whose origin is the center of each neutron star.

Then, we use Green's formula to invert the Laplacian.

◯ Coordinates and region for numerical computation.





Poisson solver : applying Green's formula.

⋆ An elliptic PDE,

$$\overset{\circ}{\Delta}\phi = S(x),$$

is written in the integral form, the Green's formula,

$$\phi(x) = -\frac{1}{4\pi}\int_V G(x,x')S(x')dV + \frac{1}{4\pi}\int_{\partial V}\Big[G(x,x')\nabla\phi(x') - \phi(x')\nabla G(x,x')\Big]\cdot dS$$

We choose the Green's funciton $G(x,x')$ without the boundary,

$$\overset{\circ}{\Delta} G(x,x') = -4\pi\delta(|x-x'|),$$

and expand in the multipoles, Legendre expansion,

$$G(x,x') = \frac{1}{|x-x'|} = \sum_{\ell=0}^{\infty} g_\ell(r,r')\sum_{m=0}^{\ell}\epsilon_m\frac{(\ell-m)!}{(\ell+m)!}P_\ell{}^m(\cos\theta)\,P_\ell{}^m(\cos\theta')\cos m(\varphi-\varphi').$$

$$g_\ell(r,r') = \frac{r_<^\ell}{r_>^{\ell+1}}, \quad r_> := \sup\{r,r'\},\ r_< := \inf\{r,r'\}$$

⋆ An iteration method is used to compute a converged solution.

$$\phi^{(N+1)} = \lambda\phi^{(\mathrm{INT})} + (1-\lambda)\phi^{(N)}, \quad \lambda:\ \text{parameter } 0.3 \sim 0.5.$$

○ Simplistic iteration algorithm.

Initialise : $\{\psi, \alpha, \beta^a, h_{ab}, \rho, \Phi; \Omega, C, R_0\}$

ex) a solution from a previous computation,
or two TOV solutions in PN orbit (BNS).

$\downarrow$

Compute source terms : S.

$\downarrow$

Substitute in a Poisson solver : $\phi^{(\mathrm{INT})} = \overset{\circ}{\Delta}{}^{-1} S$.

$\downarrow$

Update all quantities :

$$\phi^{(N+1)} = \lambda \phi^{(\mathrm{INT})} + (1-\lambda)\phi^{(N)},$$

$$\downarrow$$

Check the convergence :

$$\text{error} = \frac{2|\phi^{(N+1)} - \phi^{(N)}|}{|\phi^{(N+1)}| + |\phi^{(N)}|} < 10^{-5} \sim 10^{-6},$$

$$\downarrow$$

Not converged – go back to the second step.
Converged – Compute quantities, M, J and so on

This numerical code may be considered as a family of a scheme developed by Ostriker and Marck (1968) for Newtonian rotating star. It has been successfully extended for the GR rotating neutron star computation by Komatsu, Eriguchi, and Hachisu (1989), known as KEH code.

Related theorems :

A first law relation and a equality $M_{\mathsf{ADM}} = M_{\mathsf{K}}$

◯ First law of thermodynamics for binary systems.

(Friedman, Uryu and Shibata, PRD 2002)

◦ The first law compares two nearby equilibria having a helical K.V..

Given a family of perfect fluid spacetime,

$$Q(\lambda) := \left[g_{\alpha\beta}(\lambda), u^{\alpha}(\lambda), \rho(\lambda), s(\lambda)\right],$$

one defines the Eulerian variation of each quantities by

$$\delta Q(\lambda) := \frac{d}{d\lambda} Q(\lambda)|_{\lambda=0}$$

Lagrangian displacement ξ^{α}: Let Ψ_{λ} be a diffeo mapping each trajectory of initial fluid to a corresponding worldline of the configuration $Q(\lambda)$. The tangent $\xi^{\alpha} P$ To the path $\lambda \to \Psi_{\lambda}(P)$ can be regarded as a vector joining the fluid element at P in one configuration to a fluid element in a nearby oncfiguration.

$$\Delta Q(\lambda) := \frac{d}{d\lambda} \Psi_{-\lambda} Q(\lambda)|_{\lambda=0} = (\,\delta + \pounds_{\xi}\,) Q$$

.

We choose gauge to make k^α independent of λ.

A Noether charge Q associated with k^α is found from the action of the perfect fluid spacetime, (Wald-Iyer, Sorkin, Brown).

$$\mathcal{L} = \left(\frac{1}{16\pi} R - \epsilon\right)\sqrt{-g}.$$

$$\begin{aligned}\frac{1}{\sqrt{-g}}\delta\mathcal{L} \;&=\; -\frac{1}{16\pi}\left(G^{\alpha\beta} - 8\pi T^{\alpha\beta}\right)\delta g_{\alpha\beta} - \xi^\alpha \nabla_\beta T_\alpha{}^\beta \\ &\quad - \rho T \Delta s - \frac{h}{u^t\sqrt{-g}}\Delta(\rho u^t\sqrt{-g}) + \nabla_\alpha \Theta^\alpha\end{aligned}$$

$$\Theta^\alpha = (\epsilon + p) q^{\alpha\beta}\xi_\beta + \frac{1}{16\pi}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\beta}g^{\gamma\delta})\nabla_\beta \delta g_{\gamma\delta}.$$

Definition A Noether charge Q associated with k^α is given by,

$$Q = \oint_S Q^{\alpha\beta} dS_{\alpha\beta},$$
$$Q^{\alpha\beta} = -\frac{1}{8\pi}\nabla^\alpha k^\beta + k^\alpha B^\beta - k^\beta B^\alpha,$$

$$\left(Q_K = -\frac{1}{8\pi}\oint_S \nabla^\alpha k^\beta dS_{\alpha\beta}, \quad Q_L = \oint_S (k^\alpha B^\beta - k^\beta B^\alpha) dS_{\alpha\beta}, \right)$$

where B^α is any family of vector fields that satisfies

$$\frac{1}{\sqrt{-g}}\delta(B^\alpha\sqrt{-g}) = \Theta^\alpha,$$

We make Q finite by choosing, outside the matter,

$$\sqrt{-g}B^\alpha = \frac{\sqrt{-g}}{16\pi}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\beta}g^{\gamma\delta})|_{\lambda=0} \overset{\circ}{\nabla}_\beta \, g_{\gamma\delta}(\lambda).$$

Now, one can generalize the Bardeen-Carter-Hawking calculation to fluid with arbitrary flow.

Using Stokes theorem, Q_K is written,

$$Q_K - \sum_i Q_{Ki} = -\int_\Sigma \mathcal{L}\, d^3x + \int_\Sigma (\epsilon + p) u^\alpha u_\beta v^\beta dS_\alpha - \frac{1}{8\pi}\int_\Sigma (G^\alpha{}_\beta - 8\pi T^\alpha{}_\beta) k^\beta dS_\alpha.$$

$$Q_{Ki} = -\frac{1}{8\pi}\oint_{\mathcal{B}_i} \nabla^\alpha k^\beta dS_{\alpha\beta} = \frac{1}{8\pi}\kappa_i A_i,$$

then calculate δQ_K.

(Two identities are used, $\quad \nabla_\beta \nabla^\alpha k^\beta = R^\alpha{}_\beta\, k^\beta = \frac{1}{2} R\, k^\alpha + G_{\alpha\beta}\, k^\beta,$

and $\quad 0 = \epsilon\, k^\alpha n_\alpha + (\epsilon + p) u^\alpha u_\beta v^\beta n_\alpha + T^\alpha{}_\beta k^\beta n_\alpha.$)

For δQ_L,

$$\delta(Q_L - \sum_i Q_{Li}) = \oint_{\partial\Sigma} (k^\alpha \Theta^\beta - k^\beta \Theta^\alpha) dS_{\alpha\beta} = \int_\Sigma \nabla_\beta \Theta^\beta k^\alpha dS_\alpha - \int_\Sigma \pounds_k \Theta^\alpha dS_\alpha,$$

$$\delta Q_{Li} = \oint_{\mathcal{B}_i} (k^\alpha \Theta^\beta - k^\beta \Theta^\alpha) dS_{\alpha\beta} = -\frac{1}{8\pi}\delta\kappa_i A_i.$$

Writing

$$\bar{T} := \frac{T}{u^t}, \qquad \bar{\mu} := \frac{\mu}{u^t m_{\mathsf{B}}} = \frac{h - Ts}{u^t},$$

and

$$dM_{\mathsf{B}} := \rho u^\alpha dS_\alpha, \qquad dS := s dM_{\mathsf{B}}, \qquad dC_\alpha := h u_\alpha dM_{\mathsf{B}},$$

we have

$$\delta Q = \int_\Sigma \left[\bar{T} \Delta dS + \bar{\mu} \Delta dM_{\mathsf{B}} + v^\alpha \Delta dC_\alpha \right] + \frac{1}{8\pi} \sum_i \kappa_i \delta A_i.$$

Conservation of entropy, rest mass, and circulation of each fluid element imply

$$\delta Q = \frac{1}{8\pi} \sum_i \kappa_i \delta A_i.$$

In the post-Newtonian approximation and the related spatially conformally flat spacetimes (IWM formalism) that describe the binary neutron star systems, the metric is non-radiateive and asymptotically flat.

$$Q_{\mathsf{K}} = \frac{1}{2}M - \Omega J$$

$$\delta Q = \delta M - \Omega \delta J$$

For a change that locally preserves vorticity, baryon number and entropy,

$$\delta M = \Omega \delta J$$

Remark1 Q is independent of the 2-surface S on which it is evaluated. This is immediate for Q_K by definition. For Q, it follows from $Q = Q_K$ at $\lambda = 0$ and δQ is independent of S as shown above ($Q(\lambda) = Q(0) + \delta Q$).

Remark2 The difference $\delta(Q - \Sigma_i Q_i)$ (Q_i is black hole terms) is invariant under gauge transformations that respect the symmetry k^α.

○ Turning point stability and location of the ISCO.

The first law allows one to apply a turning point theorem (Sorkin 1981) to sequence of binary equilibria. The theorem shows that on one side of a turning point in M at fixed J or in J and fixed baryon mass M_0, the sequence is unstable.

Theorem *(Sorkin, 1981), We assume that unique Ω such that $\delta M = \Omega \delta J$, exist for any equilibriums, and that the equilibria are extrema of mass with J constant.*

Consider a one-parameter family of binary equilibrium models

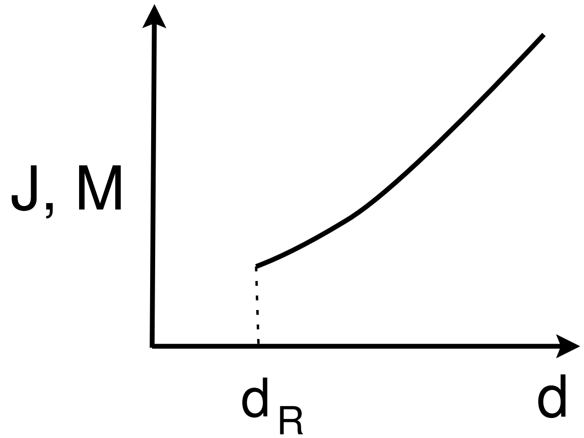
$$\mathcal{Q}(\lambda) := [g_{\alpha\beta}(\lambda), u^\alpha(\lambda), \rho(\lambda), s(\lambda)],$$

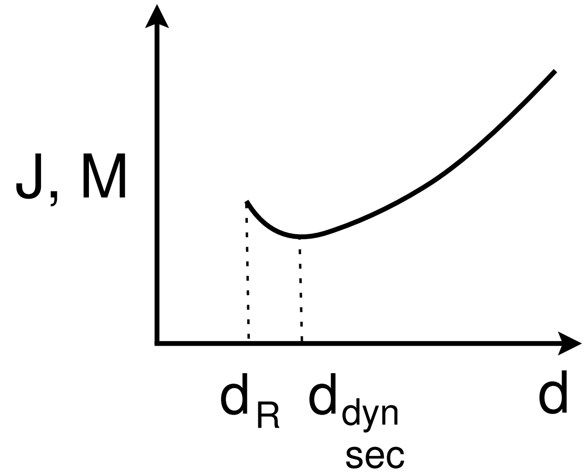
along which the Lagrangian changes $\Delta s, \Delta dM_B$, and ΔdC_α vanish. Suppose that $\dot{J} = 0$ at a point λ_0 along the sequence, and that $\dot{\Omega}\ddot{J} \neq 0$ at λ_0. Then the part of the sequence for which $\dot{\Omega}\dot{J} > 0$ is unstable for λ near λ_0.

○ Solution sequence of BNS and determination of the ISCO.

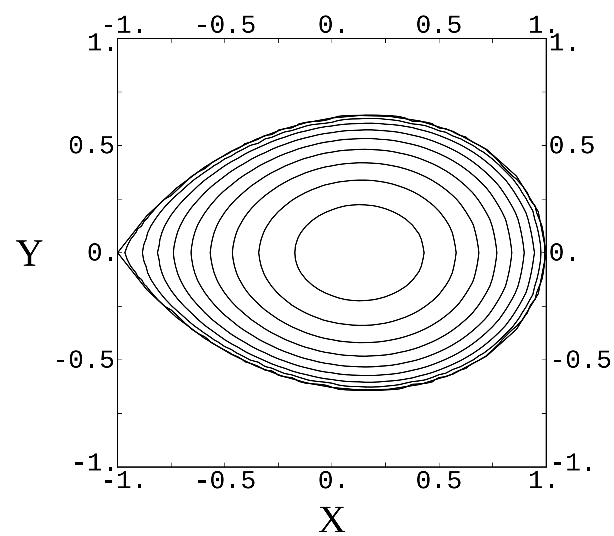
(Lai, Rasio, Shapiro 1993; Baumgarte, Cook, Scheel, Shapiro, Teukolsky 1998)

A stability of solution changes at a turning point of a sequence.





Density contour at d_R



○ M_{K}–M_{ADM} relation.

(Shibata, Uryu and Friedman, PRD 2004)

We have derived sufficient fall off behaviours of the metric and extrinsic curvature in the asymptotics to satisfy an equality $M_{\mathsf{K}} = M_{\mathsf{ADM}}$, improved results by Ashtekar and Magnon-Ashtekar, and by Beig so that we can apply the equality to the binary systems.

Beig's proof is restricted to spacetimes without black holes. Our proof relys only on the asymptotic behaviour of fields, and hence admit black holes.

From the equality $M_{\mathsf{K}} = M_{\mathsf{ADM}}$, one can derive the general relativistic virial relation, an integral

$$\int x^i \gamma_i^\mu \nabla_\nu T_\mu{}^\nu d^3x = 0$$

These relations are useful for calibrate equilibirum solutions.