Code construction – 3: Computation of source terms.

- The data of source terms is prepared on grid points of the spherical coordinate (r, θ, ϕ), and it is passed to the Poisson solver.
- Choice of grid points for the source may depend on what kind of numerical integration formula is used. Our choice is the mid-point rule.
- We have prepared the mid point grids as (hrg(irg), hthg(itg), hphig(ipg)). However, the field values $\{\psi, \alpha, \beta^a\}$ are not defined on these grids. We need to interpolate these field values on the grid points (rg(irg), thg(itg), phig(ipg)) to the mid-points (hrg(irg), hthg(itg), hphig(ipg)).

Remark: If one chose another numerical integration formula, make sure that the weight for integration is chosen correctly for non-equidistant grids.

Close look at the source terms.

- Computation goes as follows
  (1) compute $\tilde{A}_{ab}$
  (2) interpolate fluid terms to grid points.
  (3) compute sources.

- Computation of source terms include evaluations of following quantities at the mid-point grids (when the mid-point quadrature formula is applied)
  (1) the field values,
  (2) its first derivatives, and
  (3) the fluid source terms.

- Vectors and tensors have Cartesian components, such as $\{\beta_x, \beta_y, \beta_z\}$.

$$\tilde{A}_{ab} = \frac{1}{2\alpha}\left(\mathring{D}_a\tilde{\beta}_b + \mathring{D}_b\tilde{\beta}_a - \frac{2}{3}f_{ab}\mathring{D}_c\tilde{\beta}^c\right)$$

$$\rho_{\rm H} := T_{\alpha\beta}n^\alpha n^\beta = h\rho(\alpha u^t)^2 - p,$$

$$j_a := -T_{\alpha\beta}\gamma_a{}^\alpha n^\beta = h\rho\alpha(u^t)^2\psi^4\tilde{\omega}_a,$$

$$S := T_{\alpha\beta}\gamma^{\alpha\beta} = h\rho[(\alpha u^t)^2 - 1] + 3\,p.$$

$$u^t = \frac{1}{\sqrt{\alpha^2 - \omega_a\omega^a}} = \frac{1}{\sqrt{\alpha^2 - \psi^4 f_{ab}\,\tilde{\omega}^a\tilde{\omega}^b}},$$

where $\tilde{\omega}^a = \tilde{\beta}^a + \Omega\phi^a.$

$$\mathring{\Delta}\psi = -\frac{\psi^5}{8}\tilde{A}_{ab}\tilde{A}^{ab} - 2\pi\psi^5\rho_{\rm H}$$

$$\left(\mathring{\Delta}\tilde{\beta}_a + \frac{1}{3}\mathring{D}_a\mathring{D}_b\tilde{\beta}^b = -2\alpha\tilde{A}_a{}^b\mathring{D}_b\ln\frac{\psi^6}{\alpha} + 16\pi\alpha j_a\right)$$

$$\mathring{\Delta}(\alpha\psi) = \alpha\psi^5\frac{7}{8}\tilde{A}_{ab}\tilde{A}^{ab} + 2\pi\alpha\psi^5(\rho_{\rm H} + 2S)$$

$$\mathring{\Delta}B_a = \mathcal{S}_a := -2\alpha\tilde{A}_a{}^b\mathring{D}_b\ln\frac{\psi^6}{\alpha} + 16\pi\alpha j_a,$$

$$\mathring{\Delta}B = x^a\mathcal{S}_a.$$

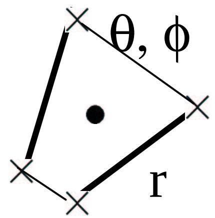
$$\tilde{\beta}_a = B_a + \frac{1}{8}\mathring{D}_a(B - x^bB_b)$$

◯ Interpolations

A few types of interpolation are necessary during the calculations of the sources.

- A linear interpolating polynomial is used for most of the cases. (This is 2nd order accurate.)
- For some cases higher order (such as 4th order) interpolating polynomial is preferable. (It is often nice to use it, when grid points to be interpolated are finer than the original grid points.)
- A linear interpolation formula is chosen to interpolate $\{\psi, \alpha, \beta^a\}$ on the grid (rg(irg), thg(itg), phig(ipg)) to the mid-points (hrg(irg), hthg(itg), hphig(ipg)). Since this has a simple geometry, the formula has a simple form.

$$hf(irg, itg, ipg) = \frac{1}{8} \sum_{i=irg-1}^{irg} \sum_{j=itg-1}^{itg} \sum_{k=ipg-1}^{ipg} f(i, j, k)$$

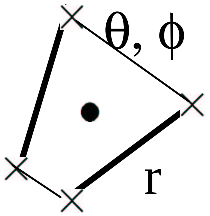


We make a subroutine, or a function for this.

Interpolation from the fluid coordinate to the field coordinate.

- A linear interpolating polynomial is used for most of the cases.
  (This is 2nd order accurate.)

- First find a position of the grid (hrg(irg), hthg(itg), hphig(ipg)) in the fluid coordinate (rf(irf), thf(itf), phif(ipf)), then apply a linear interpolating polynomials.



- See the last page of this slide set.

Interpolation from the field coordinate to the fluid coordinate.

- For a single star case, only the interpolation in r-coordinate is required.
  (θ, ϕ coordinate grids are the same for the fluid and the field.)
  Higher order interpolating polynomial may be preferable.

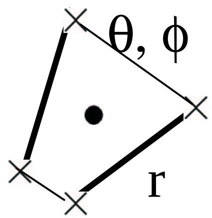
◯ Gradient on the field coordinate grids.

- To compute the gradients of {ψ, α, β[a] } at the mid-points (hrg(irg), hthg(itg), hphig(ipg)) using the values at grids (rg(irg), thg(itg), phig(ipg)), the following procedure is used.

$$\partial_r f(irg, itg, ipg) = \frac{1}{4} \sum_{j=itg-1}^{itg} \sum_{k=ipg-1}^{ipg} \frac{f(irg, j, k) - f(irg-1, j, k)}{drg(irg)}$$

$$\partial_\theta f(irg, itg, ipg) = \frac{1}{4} \sum_{i=irg-1}^{irg} \sum_{k=ipg-1}^{ipg} \frac{f(i, itg, k) - f(i, itg-1, k)}{dthg(itg)}$$

$$\partial_\phi f(irg, itg, ipg) = \frac{1}{4} \sum_{i=irg-1}^{irg} \sum_{j=itg-1}^{itg} \frac{f(i, j, ipg) - f(i, j, ipg-1)}{dphig(ipg)}$$



Then “rotate”,

$$\begin{pmatrix} \partial_x f \\ \partial_y f \\ \partial_z f \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \partial_r f \\ \frac{1}{r}\partial_\theta f \\ \frac{1}{r\sin\theta}\partial_\phi f \end{pmatrix}$$

Evaluated at mid-point (hrg(irg), hthg(itg), hphig(ipg)) .

- Source terms of each equations are calculated in a separated subroutine.

subroutine iteration

call extrinsic_curvature

call interpolate_fluid_to_grav

call sourceterm_hc(sou_psi)

call poisson_solver(sou_psi,pots)

call update_grfield(pot,psi)

call sourceterm_spatialtr(sou_alps)

call poisson_solver(sou_alps,pots)

call update_grfield(pot,alps)

call sourceterm_momc(sou_bxyzs)

call poisson_solver(sou_bx,potx)

call poisson_solver(sou_by,poty)

call poisson_solver(sou_bz,potz)

call poisson_solver(sou_bs,pots)

call update_grfield() $\times$ 4

call compute_shift

- Then, computation of the fluid comes here.

end subroutine iteration

subroutine interpolate_fluid_to_grav

- Definitions of the coordinates for the gravitational and the fluid matter are given in the other slides.

$$hR(itf,ipf) = \frac{1}{4} \sum_{i=itf-1}^{itf} \sum_{j=ipf-1}^{ipf} R(i,j)$$

This formula gives a neutron star radius corresponds to the mid-point hthg(itg), hphig(ipg), itf = itg, ipf = ipg.

Therefore hrg(irg)/hR(itg,ipg) is the radial coordinate of the mid-point of the gravitational coordinate grid in the fulid coordinate grid point.

(This is the same as mapping a mid-point of gravitaional coordinate grids to the fluid coordinate grids.)

Use values of f(i,j,k) at the coordinate (rf(i), thf(j), phif(k)) at
i = irf, irf-1, j = itf, itf-1, k = ipf, ipf-1 (8 points), and make 3D linear interpolation to the (mid) point (r, θ,ϕ) = (hrg(irg)/hR(itg,ipg), hthg(itg), hphig(ipg)), where hthg(itg) = hthf(itf), hphig(ipg) = hphif(ipf).

end subroutine interpolate_fluid_to_grav