Mass and Angular momentum formulas

- Various quantities are calculated for converged solutions, which are used to understand the property of each solution.
- These includes the total baryon mass, ADM mass, Komar mass, ADM angular momentum.

- Several definitions for the horizon mass are also used for the black hole spacetime.

Formula for the total baryon mass.

$$M_0 := \int_\Sigma \rho\, u^\alpha dS_\alpha = \int_\Sigma \rho\, u^\alpha \nabla_\alpha t \sqrt{-g} d^3x$$
$$= \int_\Sigma \alpha\, u^t \sqrt{\gamma} d^3x = \int_\Sigma \alpha\, u^t \psi^6 \sqrt{\tilde{\gamma}} d^3x$$

For conformal flat spatial slice, $\sqrt{\tilde{\gamma}} d^3x = \sqrt{f} d^3x = r^2 \sin\theta dr d\theta d\phi$

Formula for the ADM mass. On an asymptotically flat spacelike hypersurface,

$$M_{\mathrm{ADM}} := \frac{1}{16\pi}\int_\infty \left(f^{ac}f^{bd} - f^{ab}f^{cd}\right)\mathring{D}_b\gamma_{cd}\,dS_a$$
$$= \frac{1}{16\pi}\int_\infty \left(\gamma^{ac}\gamma^{bd} - \gamma^{ab}\gamma^{cd}\right)\mathring{D}_b\gamma_{cd}\,dS_a$$
$$= \underbrace{\frac{1}{16\pi}\int_\infty \left(\tilde\gamma^{ac}\tilde\gamma^{bd} - \tilde\gamma^{ab}\tilde\gamma^{cd}\right)\mathring{D}_b\tilde\gamma_{cd}}\,dS_a - \frac{1}{2\pi}\int_\infty \tilde\gamma^{ab}\mathring{D}_b\psi\,dS_a$$

For $\tilde\gamma = f$ the first term vanishes.

In the flat asymptotics, r $\rightarrow \infty$, $\left\{\begin{array}{l} d\mathring{S}_a = \sqrt{f}\nabla_a r d^2x \\ d\tilde{S}_a = \sqrt{\tilde\gamma}\nabla_a r d^2x \\ dS_a = \sqrt{\gamma}\nabla_a r d^2x \end{array}\right\}$ all agree.

$$M_{\mathrm{ADM}} := -\frac{1}{2\pi}\int_\infty \tilde{D}^a\psi\,d\tilde{S}_a = -\frac{1}{2\pi}\int_\Sigma \tilde\Delta\psi\,d\tilde{S}$$
$$:= \frac{1}{2\pi}\int_\Sigma\left[-\frac{1}{8}\psi\overset{3}{\tilde{R}} + \frac{1}{8}\psi^5\left(\tilde{A}_{ab}\tilde{A}^{ab} - \frac{2}{3}K^2\right) + 2\pi\psi^5\rho_{\mathrm{H}}\right]\sqrt{\tilde\gamma}d^3x$$

For confomally flat slice, $M_{\mathrm{ADM}} := -\frac{1}{2\pi}\int_\infty \mathring{D}^a\psi\,d\mathring{S}_a = -\frac{1}{2\pi}\int_\Sigma \mathring{\Delta}\psi\,d\mathring{S}$

(It is better integrate a fluid term on the fluid coordinate grids.) $:= \frac{1}{2\pi}\int_\Sigma\left[\frac{1}{8}\psi^5\tilde{A}_{ab}\tilde{A}^{ab} + 2\pi\psi^5\rho_{\mathrm{H}}\right]\sqrt{f}d^3x$

Formula for the Komar mass. For an asymptotically timelike Killing field t^α,

$$M_\mathsf{K} := -\frac{1}{4\pi}\int_\infty \nabla^\alpha t^\beta \, dS_{\alpha\beta} = \frac{1}{4\pi}\int_\infty \left(D^a\alpha - K^a{}_b\beta^b\right) dS_a = \frac{1}{4\pi}\int_\infty D^a\alpha \, dS_a$$

$$= \frac{1}{4\pi}\int_\Sigma \Delta\alpha \, d\Sigma = \frac{1}{4\pi}\int_\Sigma \left[\alpha \tilde{A}_{ab}\tilde{A}^{ab} + 4\pi\alpha\left(\rho_\mathsf{H} + S\right)\right]\psi^6\sqrt{\tilde{\gamma}}d^3x$$

$$dS_\alpha = \nabla_\alpha t\sqrt{-g}d^3x = -n_\alpha\sqrt{\gamma}d^3x = -n_\alpha d\Sigma$$

For confomally flat slice, $M_\mathsf{K} := \frac{1}{4\pi}\int_\Sigma \left[\alpha \tilde{A}_{ab}\tilde{A}^{ab} + 4\pi\alpha\left(\rho_\mathsf{H} + S\right)\right]\psi^6\sqrt{f}d^3x$

(It is better integrate a fluid term on the fluid coordinate grids.)

$M_\mathsf{K} = M_\mathsf{ADM}$ for the asymptotically flat spacetime.

Formula for the angular momentum.

$$J \;:=\; -\frac{1}{8\pi}\int_\infty \pi^a{}_b\phi^b\,dS_a \;=\; \frac{1}{8\pi}\int_\infty K^a{}_b\phi^b\,dS_a$$
$$\;=\; \frac{1}{8\pi}\int_\Sigma D_a(K^a{}_b\phi^b)\,dS$$

For confomally flat slice, $J = \frac{1}{8\pi}\int_\Sigma 8\pi j_a\phi^a\psi^6\sqrt{f}d^3x$