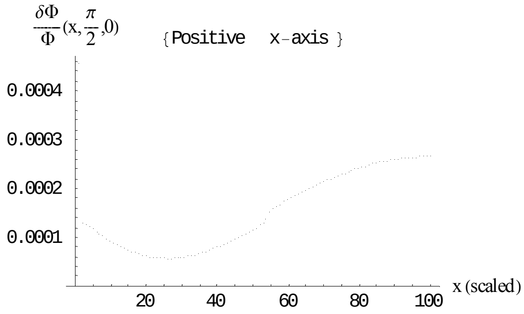
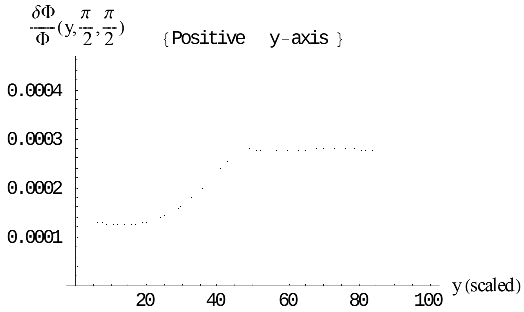
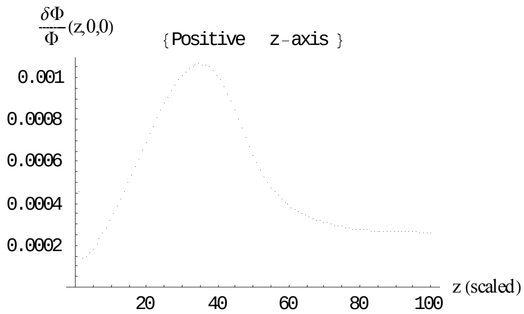
1. Poisson Solver (Trapezoidal Rule)

A spherical star is shifted slightly from the origin, and is centered on the positive x-axis. In the first plot, the center corresponds to the $x = 5$ point and the radius to $\Delta x = 50$ points (the actual radius is different and is not equidistant; I am just plotting the points equidistantly here). I compare the analytical to the numerical solution, and plot the relative error $\delta\Phi / \Phi$.







In the above, we directly applied the trapezoidal rule for the θ integration:

$$\int_j^{j+1} S(\theta) P_l^m(\cos\theta)\sin\theta\,\mathrm{d}\,\theta \equiv \int_j^{j+1} f(\theta)\,\mathrm{d}\,\theta \simeq [f(\theta_{j+1}) + f(\theta_j)]\frac{\theta_{j+1} - \theta_j}{2}$$

Because $\Delta\theta$ is constant but $P_l^m(\cos\theta)\sin\theta\Delta\theta$ is not, the integration is not isotropic. It thus creates spurious multipoles and reduces the accuracy of the result, especially near the *z*-axis.

2. Poisson Solver (Absorbing the Legendre polynomials in the differential)

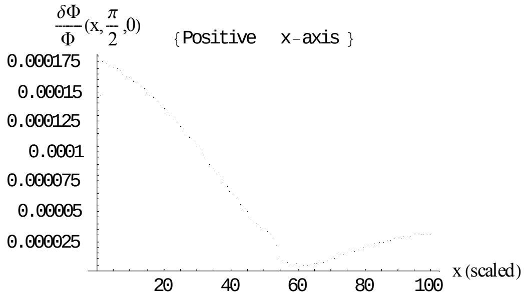
The poor integration in the θ coordinate described previously treats space in an anisotropic way. To improve the accuracy, we can absorb $\sin\theta$ or even $P_l^m(\cos\theta)$ inside the differential, and then apply the trapezoidal rule:

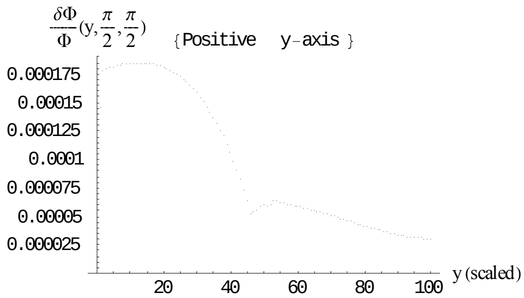
$$\int_j^{j+1} S(\theta) P_l^m(\cos\theta)\sin\theta\,\mathrm{d}\,\theta \equiv \int_j^{j+1} S(\theta)\,\mathrm{d}(IP_l^m(\cos\theta)) \simeq [S(\theta_{j+1})+S(\theta_j)]\frac{IP_l^m(\cos\theta_{j+1})-IP_l^m(\cos\theta_j)}{2}$$

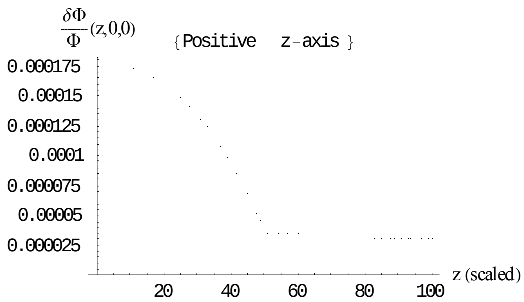
where $IP_l^m(\chi)$ denotes the indefinite integral

$$IP_l^m(\chi) \equiv \int P_l^m(\chi) d\chi$$

which can be computed analytically. Here are the results.







We see that the maximum error is reduced by a factor of 2 in the *x* and *y*-axes, and by a factor of 6 in the *z*-axis, since the unphysical multipoles are now eliminated. The integration is now isotropic. The improvement is not so dramatic (only a factor of 2) when the star is completely off the origin, so it does not touch the z-axis. But the improvement is quite significant (about a factor of 10 near the *z*-axis) when the star is centered at the origin, whence we integrate over the star density near the *z*-axis.

The only cost of the above method is that we have to compute and store values for the integrals of the Legendre Polynomials for all the polar angles at the grid points, but it does not affect the speed of the Poisson solver. It improves the accuracy, provided that the Legendre polynomials and their integrals are computed accurately. For example, the following form for $P_{10}^2(x)$ is not accurate

$$-\frac{3003}{256}\left(-1+x^2\right)\left(-3+225\,x^2-2550\,x^4+9690\,x^6-14535\,x^8+7429\,x^{10}\right)$$

because it causes underflow/overflow for small or large x. But the following form

$$-\frac{3003}{256}\,(-1+x)\,(1+x)\,\left(-3+x^2\left(225+17\,x^2\left(-150+x^2\left(570-x^2\left(855-437\,x^2\right)\right)\right)\right)\right)$$

avoids this problem and is preferred. The same is true for the integrals of the polynomials.

3. Analytical Solution

The analytical solution that we used to compare to the numerical, was obtained as follows. A spherical star, centered at the origin, with density

$$\rho(\mathbf{r}) = \rho_c \frac{\sin(\pi r / R)}{\pi r / R}$$

has a mass

$$M = \frac{4}{\pi}\rho_c R^3$$

and creates a Newtonian potential

$$\Phi(\mathbf{r}) = \begin{cases} -\dfrac{GM}{r}, & r \geq R \\ -\dfrac{GM}{R}\left(1 + \dfrac{\sin(\pi r / R)}{\pi r / R}\right), & r < R \end{cases}$$

If the star is centered at $\mathbf{r}_0 = (r_0, \theta_0, \varphi_0)$, then the above formulas hold if, in their right hand side, we replace r by

$$|\mathbf{r} - \mathbf{r}_0| = \{r^2 + r_0^2 - 2rr_0[\cos\theta\cos\theta_0 + \cos(\varphi - \varphi_0)\sin\theta\sin\theta_0]\}^{1/2}$$

In our case we chose $\mathbf{r}_0 = (x_0, \pi / 2, 0)$ so that the star is centered at the positive *x*-axis. A small program that calculates the analytical solution automatically for any given integrable spherical density distribution, is uploaded at http://pantherfile.uwm.edu/markakis/NR/PoissonSolverAnalytical.nb for everyone's use.