A collection of equations for helically symmetric perfect fluid spacetimes

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Equations for helically symmetric spacetime that used for a nuemrical code are collected. "Section V Asymptotics" is not for this note.

I. FOLIATION

We consider globally hyperbolic spacetimes $\mathcal{M}, g_{\alpha\beta}$ with perfect fluid sources. Our formulation is applied to spacetimes written in $3+1$ form, with metric γ_{ab} on Cauchy surfaces Σ.

Let t^α be a timelike vector field transverse to a Cauchy surface $\Sigma = \Sigma_0$ and let $\Sigma_t = \chi_t(\Sigma_0)$ the family of slices with the diffeo χ_t generated by t^α. Let n^α be the future-pointing unit normal to this foliation. The projection 4-tensor $\gamma_{\alpha\beta}$ orthogonal to n_α, is defined as

$$\gamma_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta, \tag{1}$$

which is associated with three metric $\gamma_{ab}(t)$ on the spatial slices Σ_t. Any spatial tensors on Σ_t can be identified with spacetime tensors which are orthogonal to n^α on all of their indices. The timelike vector t^α transverse to a family of spacelike hypersurfaces Σ_t, which allows to introduce a nonvanishing lapse α and a shift β^α that relate $\partial_t \equiv t^\alpha$ to n^α in the usual way,

$$t^\alpha = \alpha n^\alpha + \beta^\alpha, \tag{2}$$

where the shift is apatial $\beta^\alpha \nabla_\alpha t = 0$, and the timelike vector is normalized by $t^\alpha \nabla_\alpha t = 1$. The future-pointing unit normal to the hypersuface, n^α, is then written

$$n_\alpha = -\alpha \nabla_\alpha t. \tag{3}$$

Then, in a chart $\{t, x^i\}$ for which Σ_t is a t =constant surface, the metric $g_{\alpha\beta} = \gamma_{\alpha\beta} - n_\alpha n_\beta$ has the form

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \tag{4}$$

With D_a the covariant derivative associated with the spatial metric γ_{ab}, the extrinsic curvature of Σ_t is given by

$$K_{ab} = -\frac{1}{2}\pounds_n \gamma_{ab} = -\frac{1}{2\alpha}\partial_t \gamma_{ab} + \frac{1}{2\alpha}\pounds_\beta \gamma_{ab}, \tag{5}$$

where γ_{ab}, $\partial_t \gamma_{ab}$, and $\pounds_n \gamma_{ab}$ are the pullback to Σ_t of $\gamma_{\alpha\beta}$, $\pounds_t \gamma_{\alpha\beta}$, and $\pounds_n \gamma_{\alpha\beta}$, respectively.

We introduce a helical vector, time translation vector in the rotating frame. Let k^α be the helical vector field transverse to a Cauchy surface $\Sigma = \Sigma_0$ and let $\Sigma_{\tilde t} = \chi_{\tilde t}(\Sigma_0)$ the family of slices with the diffeo $\chi_{\tilde t}$ generated by k^α. The helical vector k^α may not necessary be everywhere timelike: in fact, it tends to be spacelike outside of the light cylinder to spatial infinity. It is, however, always transverse to a family of spacelike hypersurfaces $\Sigma_{\tilde t}$, which allows to introduce a nonvanishing lapse α and a shift β^α that relate $\partial_{\tilde t} \equiv k^\alpha$ to n^α in the usual way,

$$k^\alpha = \alpha n^\alpha + \omega^\alpha, \tag{6}$$

where ω^α is the rotating shift. The spatial vector ω^α relates to the usual shift β^α by

$$\omega^\alpha = \beta^\alpha + \Omega \phi^\alpha, \tag{7}$$

where, ϕ^α is the generater of a circular coordinate ϕ with a parameter length 2π, and a constant Ω is the angular velocity of the frame rotation. The generator is also spatial, $\phi^\alpha n_\alpha = 0$. One can relate the generator of time coordinate in the inertial frame to the helical vector as

$$k^\alpha = t^\alpha + \Omega \phi^\alpha. \tag{8}$$

The helical vector is normalized by $k^\alpha \nabla_\alpha t = 1$. Then, in a chart $\{\tilde t, x^i\}$ for which $\Sigma_{\tilde t}$ is a $\tilde t$ =constant surface, the metric $g_{\alpha\beta} = \gamma_{\alpha\beta} - n_\alpha n_\beta$ has the form

$$ds^2 = -\alpha^2 d\tilde t^2 + \gamma_{ij}(dx^i + \omega^i d\tilde t)(dx^j + \omega^j d\tilde t). \tag{9}$$

With D_a the covariant derivative associated with the spatial metric γ_{ab}, the extrinsic curvature of $\Sigma_{\tilde{t}}$ is given by

$$K_{ab} = -\frac{1}{2}\pounds_n \gamma_{ab} = -\frac{1}{2\alpha}\partial_{\tilde{t}}\gamma_{ab} + \frac{1}{2\alpha}\pounds_\omega \gamma_{ab}, \tag{10}$$

where γ_{ab}, $\partial_{\tilde{t}}\gamma_{ab}$, and $\pounds_n\gamma_{ab}$ are the pullback to $\Sigma_{\tilde{t}}$ of $\gamma_{\alpha\beta}$, $\pounds_{\tilde{t}}\gamma_{\alpha\beta}$, and $\pounds_n\gamma_{\alpha\beta}$, respectively.

Hereafter $\pounds_n$ operating to spatial tensors such as K_{ab} is understood as follows.

$$\pounds_n K_{ab} = \frac{1}{\alpha}\pounds_{\alpha n} K_{ab} = \frac{1}{\alpha}\left(\partial_t - \pounds_\beta\right) K_{ab} = \frac{1}{\alpha}\left(\partial_{\tilde{t}} - \pounds_\omega\right) K_{ab} \tag{11}$$

where $\partial_t K_{ab}$ is a pull back of $\pounds_t K_{\alpha\beta}$ onto Σ_t, and $\partial_{\tilde{t}} K_{ab}$ is a pull back of $\pounds_k K_{\alpha\beta}$ onto $\Sigma_{\tilde{t}}$. The above is collect becuase $K_{\alpha\beta}$ is spatial.

A. Helical symmetry

For k^α a symmetry vector,

$$\pounds_k g_{\alpha\beta} = 0, \tag{12}$$

implies

$$\pounds_k \alpha = \pounds_k \omega^\alpha = \pounds_k \omega_\alpha = \pounds_k \gamma_{\alpha\beta} = \pounds_k \gamma^{\alpha\beta} = \pounds_k K_{\alpha\beta} = \pounds_k K^{\alpha\beta} = 0 \tag{13}$$

where $\pounds_k$ operating to the spatial quantities are understood as $\pounds_k = \partial_{\tilde{t}}$. Hereafter k^α may not be a symmetry unless specified.

In the helically symmetric spacetime, the extrinsic curvauture is written

$$K_{ab} = \frac{1}{2\alpha}\pounds_\omega \gamma_{ab}. \tag{14}$$

B. Waveless formulation

Waveless formulation restrict the time derivative of the conformal metric $\tilde{\gamma}_{ab} := \psi^{-4}\gamma_{ab}$ as

$$\partial_t \tilde{\gamma}^{ab} = O(r^{-3}), \tag{15}$$

and we choose a stronger condition,

$$\partial_t \tilde{\gamma}^{ab} = 0, \tag{16}$$

which also imply $\partial_t \tilde{\gamma}_{ab} = 0$. For the stronger condition, the extrinsic curvature is written,

$$\begin{aligned} K_{ab} = -\frac{1}{2}\pounds_n \gamma_{ab} &= -\frac{1}{2\alpha}\partial_t \gamma_{ab} + \frac{1}{2\alpha}\pounds_\beta \gamma_{ab} &(17)\\ &= \frac{1}{2\alpha}\pounds_\beta \gamma_{ab} - \frac{1}{3\alpha}\gamma_{ab}\left(\frac{\tilde{\gamma}}{\gamma}\right)^{\frac{1}{2}}\partial_t\left(\frac{\gamma}{\tilde{\gamma}}\right)^{\frac{1}{2}} &(18)\\ &= \frac{1}{2\alpha}\pounds_\beta \gamma_{ab} - \frac{1}{3\alpha}\gamma_{ab}\frac{1}{\sqrt{\gamma}}\partial_t\sqrt{\gamma} &(19)\end{aligned}$$

where we used

$$\gamma_{ab} = \left(\frac{\gamma}{\tilde{\gamma}}\right)^{1/3}\tilde{\gamma}_{ab} \tag{20}$$

If we impose helical symmetry (stationary condition in the rotating frame) to the other variables, and hence $\pounds_k \gamma = 0$, we have

$$\begin{aligned} K_{ab} &= \frac{1}{2\alpha}\pounds_\beta \gamma_{ab} + \frac{1}{3\alpha}\gamma_{ab}\frac{1}{\sqrt{\gamma}}\pounds_{\Omega\phi}\sqrt{\gamma} &(21)\\ &= \frac{1}{2\alpha}\pounds_\beta \gamma_{ab} + \frac{1}{3\alpha}\gamma_{ab}\, D_c(\Omega\phi^c). &(22)\end{aligned}$$

II. CONFORMAL DECOMPOSITION, GAUGE AND OTHER CONDITIONS

A. Conformal decomposition

Conformal decomposition: ψ a conformal factor (scalar function),

$$\gamma_{ab} = \psi^4 \tilde{\gamma}_{ab}. \tag{23}$$

Flat metric: f_{ab}

$$\tilde{\gamma}_{ab} = f_{ab} + h_{ab} \quad \text{and} \quad \tilde{\gamma}^{ab} = f^{ab} + h^{ab}. \tag{24}$$

$$\gamma_{ac}\, \gamma^{cb} = \delta_a{}^b \tag{25}$$
$$\tilde{\gamma}_{ac}\, \tilde{\gamma}^{cb} = \delta_a{}^b \tag{26}$$
$$f_{ac}\, f^{cb} = \delta_a{}^b \tag{27}$$
$$h_{ac}\, h^{cb} \neq \delta_a{}^b \tag{28}$$
$$\tag{29}$$

Determinants

$$\gamma := \det \gamma_{ab} \tag{30}$$
$$\tilde{\gamma} := \det \tilde{\gamma}_{ab} \tag{31}$$
$$f := \det f_{ab} \tag{32}$$
$$\tag{33}$$

Condition for the conformal decomposition

$$\tilde{\gamma} = f. \tag{34}$$

Therefore

$$\sqrt{\gamma} = \psi^6 \sqrt{\tilde{\gamma}} = \psi^6 \sqrt{f}. \tag{35}$$

Derivatives :

$$D_a \quad \text{is compatible with} \quad \gamma_{ab} \tag{36}$$
$$\tilde{D}_a \quad \text{is compatible with} \quad \tilde{\gamma}_{ab} \tag{37}$$
$$\mathring{D}_a \quad \text{is compatible with} \quad f_{ab} \tag{38}$$
$$\partial_a \quad \text{is} \quad \mathring{D}_a \quad \text{in Cartesian cooridnate.} \tag{39}$$

For example, for a spatial vector ω^a, with $\tilde{\gamma} = f$

$$\frac{1}{\sqrt{\gamma}} \partial_a (\sqrt{\gamma}\omega^a) = D_a \omega^a \tag{40}$$
$$= \frac{1}{\psi^6 \sqrt{\tilde{\gamma}}} \partial_a (\psi^6 \sqrt{\tilde{\gamma}} \omega^a) = \frac{1}{\psi^6} \tilde{D}_a (\psi^6 \omega^a) \tag{41}$$
$$= \frac{1}{\psi^6 \sqrt{f}} \partial_a (\psi^6 \sqrt{f} \omega^a) = \frac{1}{\psi^6} \mathring{D}_a (\psi^6 \omega^a) \tag{42}$$

B. Connection coefficients

$$D_b X^a = \tilde{D}_b X^a + \tilde{C}^a_{bc} X^c \tag{43}$$
$$\tilde{D}_b X^a = \mathring{D}_b X^a + C^a_{bc} X^c \tag{44}$$

$$\tilde{C}^a_{bc} = \frac{1}{2}\gamma^{ad}(\tilde{D}_b \gamma_{dc} + \tilde{D}_c \gamma_{bd} - \tilde{D}_d \gamma_{bc}) = 2\,(\tilde{\gamma}^a{}_c \tilde{D}_b \ln\psi + \tilde{\gamma}^a{}_b \tilde{D}_c \ln\psi - \tilde{\gamma}_{bc}\tilde{\gamma}^{ad}\tilde{D}_d \ln\psi) \tag{45}$$
$$C^a_{bc} = \frac{1}{2}\tilde{\gamma}^{ad}(\mathring{D}_b \tilde{\gamma}_{dc} + \mathring{D}_c \tilde{\gamma}_{bd} - \mathring{D}_d \tilde{\gamma}_{bc}) = \frac{1}{2}\tilde{\gamma}^{ad}(\mathring{D}_b h_{dc} + \mathring{D}_c h_{bd} - \mathring{D}_d h_{bc}) \tag{46}$$

With a condition $\gamma = f$

$$C^c_{cb} = \frac{1}{2}\tilde{\gamma}^{cd}\mathring{D}_b \tilde{\gamma}_{cd} = \frac{1}{\sqrt{\tilde{\gamma}}}\mathring{D}_b \sqrt{\tilde{\gamma}} = \frac{1}{\sqrt{f}}\mathring{D}_b \sqrt{f} = 0 \tag{47}$$

Change index for a reference

$$\tilde{C}^c_{ab} = \frac{1}{2}\gamma^{cd}(\tilde{D}_a \gamma_{db} + \tilde{D}_b \gamma_{ad} - \tilde{D}_d \gamma_{ab}) = 2\,(\tilde{\gamma}^c{}_b \tilde{D}_a \ln\psi + \tilde{\gamma}^c{}_a \tilde{D}_b \ln\psi - \tilde{\gamma}_{ab}\tilde{\gamma}^{cd}\tilde{D}_d \ln\psi) \tag{48}$$
$$C^c_{ab} = \frac{1}{2}\tilde{\gamma}^{cd}(\mathring{D}_a \tilde{\gamma}_{db} + \mathring{D}_b \tilde{\gamma}_{ad} - \mathring{D}_d \tilde{\gamma}_{ab}) = \frac{1}{2}\tilde{\gamma}^{cd}(\mathring{D}_a h_{db} + \mathring{D}_b h_{ba} - \mathring{D}_d h_{ab}) \tag{49}$$

1. Other formula

$$\tilde{\gamma}^{bd} C^a_{dc} = \frac{1}{2}\tilde{\gamma}^{bd}\tilde{\gamma}^{ae}(\mathring{D}_d \tilde{\gamma}_{ec} + \mathring{D}_c \tilde{\gamma}_{de} - \mathring{D}_e \tilde{\gamma}_{dc}) \tag{50}$$

For the divergence of the tensor A_{ab}

$$D_b A_a{}^b = \tilde{D}_b A_a{}^b + \tilde{C}^b_{bd} A_a{}^d - \tilde{C}^d_{ba} A_d{}^b \tag{51}$$
$$= \frac{1}{\sqrt{\gamma}}\tilde{D}_b\left(\sqrt{\gamma} A_a{}^b\right) - \tilde{C}^c_{ba} A_c{}^b \tag{52}$$
$$= \frac{1}{\sqrt{\gamma}}\tilde{D}_b\left(\sqrt{\gamma} A_a{}^b\right) - 2 A_b{}^b \tilde{D}_a \ln\psi \tag{53}$$

C. Derivatives of densitized quantities

For an operator ∂ that follows Leibnitz rule, $\partial\gamma = \gamma\gamma^{ab}\partial\gamma_{ab}$,

$$\frac{1}{\gamma}D_c\gamma = \gamma^{ab}D_c\gamma_{ab} = 0. \tag{54}$$

$$D_a\sqrt{\gamma} = 0, \tag{55}$$

$$\tilde{D}_a\sqrt{\tilde{\gamma}} = 0 \tag{56}$$

$$\mathring{D}_a\sqrt{f} = 0 \tag{57}$$

$$\frac{1}{\sqrt{\gamma}}\tilde{D}_a\sqrt{\gamma} = \frac{1}{\sqrt{\gamma}}\tilde{D}_a(\psi^6\sqrt{\tilde{\gamma}}) = \frac{1}{\psi^6}\tilde{D}_a\psi^6, \tag{58}$$

$$\frac{1}{\sqrt{\gamma}}\mathring{D}_a\sqrt{\gamma} = \frac{1}{\sqrt{\gamma}}\mathring{D}_a(\psi^6\sqrt{\tilde{\gamma}}) = \frac{1}{\psi^6}\mathring{D}_a\psi^6 + \frac{1}{\sqrt{\tilde{\gamma}}}\mathring{D}_a\sqrt{\tilde{\gamma}}, \tag{59}$$

For $\tilde{\gamma} = f$,

$$\mathring{D}_a\sqrt{\tilde{\gamma}} = 0 \tag{60}$$

$$\frac{1}{\sqrt{\gamma}}\mathring{D}_a\sqrt{\gamma} = \frac{1}{\psi^6}\mathring{D}_a\psi^6, \tag{61}$$

$$\frac{1}{\psi^6}\tilde{D}_a\psi^6 = \frac{1}{\sqrt{\gamma}}\tilde{D}_a\sqrt{\gamma}, \tag{62}$$

Also for $\tilde{\gamma} = f$, since

$$\tilde{D}_c\gamma_{ab} = \mathring{D}_c\gamma_{ab} - C^d_{ca}\gamma_{db} - C^d_{cb}\gamma_{ad}, \tag{63}$$

$$\gamma^{ab}\tilde{D}_c\gamma_{ab} = \gamma^{ab}\mathring{D}_c\gamma_{ab} - 2C^a_{ca} = \gamma^{ab}\mathring{D}_c\gamma_{ab}, \tag{64}$$

we have

$$\frac{1}{\sqrt{\gamma}}\tilde{D}_c\sqrt{\gamma} = \frac{1}{\sqrt{\gamma}}\mathring{D}_c\sqrt{\gamma}, \tag{65}$$

$$\frac{1}{\psi^6}\tilde{D}_c\psi^6 = \frac{1}{\psi^6}\mathring{D}_c\psi^6, \tag{66}$$

and hence,

$$\tilde{D}_c\psi = \mathring{D}_c\psi \tag{67}$$

D. Gauge conditions

(1)Gauge: maximal slicing

$$K = 0. \tag{68}$$

(2)Gauge: generalized Dirac gauge condition

$$F^a := \mathring{D}_b \tilde{\gamma}^{ab} = \mathring{D}_b h^{ab} = 0. \tag{69}$$

1. *Useful formula*

(1) From the Dirac gauge,

$$\tilde{\gamma}^{bc} \mathring{D}_c \tilde{\gamma}_{ab} = \tilde{\gamma}^{bc} \mathring{D}_c h_{ab} = 0. \tag{70}$$

since

$$0 = \mathring{D}_b \tilde{\gamma}^{ab} = \mathring{D}_b (\tilde{\gamma}^{ac} \tilde{\gamma}^{bd} \tilde{\gamma}_{cd}) = \mathring{D}_b (\tilde{\gamma}^{ac}) \tilde{\gamma}^{bd} \tilde{\gamma}_{cd} + \tilde{\gamma}^{ac} \tilde{\gamma}^{bd} \mathring{D}_b \tilde{\gamma}_{cd} \tag{71}$$

(2) From the Dirac gauge and $\tilde{\gamma} = f$ condition

$$\tilde{\gamma}^{bc} C^a_{bc} = \frac{1}{2} \tilde{\gamma}^{bc} \tilde{\gamma}^{ad} (\mathring{D}_b \tilde{\gamma}_{dc} + \mathring{D}_c \tilde{\gamma}_{bd} - \mathring{D}_d \tilde{\gamma}_{bc}) = 0. \tag{72}$$

(3) From the above relation,

$$\tilde{\gamma}^{bc} \mathring{D}_d C^a_{bc} = \mathring{D}_d (\tilde{\gamma}^{bc} C^a_{bc}) - C^a_{bc} \mathring{D}_d \tilde{\gamma}^{bc} \tag{73}$$

2. *1+log slicing*

$$\pounds_t \alpha = -2\alpha K, \quad \text{or} \quad \pounds_t \alpha = -2\alpha (K - K_0). \tag{74}$$

E. Extrinsic curvature under the helical symmetry and the gauge

For trace part of K_{ab}, the maximal slicing implies

$$K = -\frac{1}{2}\gamma^{ab}\pounds_n\gamma_{ab} = \frac{1}{\alpha}\left(-\frac{1}{\sqrt{\gamma}}\pounds_k\sqrt{\gamma} + D_a\omega^a\right) = 0. \tag{75}$$

$\pounds_k f_{ab} = 0$ with $\tilde{\gamma} = f$,

$$K = \frac{1}{\alpha\psi^6}\left[-\pounds_k\psi^6 + \mathring{D}_a(\psi^6\omega^a)\right] = \frac{1}{\alpha}\left(-6\pounds_k \ln\psi + D_a\omega^a\right) = 0. \tag{76}$$

For k^α a symmetry, $\pounds_k\gamma_{ab} = \pounds_k f_{ab} = 0$ implies $\pounds_k\psi = 0$ and hence $\pounds_k\tilde{\gamma}_{ab} = 0$. Then, one may be allowed to impose

$$\frac{6}{\psi}\pounds_k\psi = D_a\omega^a, \quad \pounds_k\psi = 0 \quad \text{and} \quad D_a\omega^a = 0 \tag{77}$$

or

$$\pounds_k\psi^6 = \tilde{D}_a(\psi^6\omega^a), \quad \pounds_k\psi = 0 \quad \text{and} \quad \tilde{D}_a(\psi^6\omega^a) = 0. \tag{78}$$

$$\pounds_k\psi^6 = \tilde{D}_a(\psi^6\omega^a) \ \text{ is equivalent to } \ \frac{6}{\psi}\pounds_{\alpha n}\psi = \tilde{D}_a\omega^a. \tag{79}$$

Trace free part of K_{ab}, i.e. $A_{ab} = K_{ab} - \frac{1}{3}\gamma_{ab}K$,

$$A_{ab} = -\frac{1}{2}\left(\pounds_n\gamma_{ab} - \frac{1}{3}\gamma_{ab}\gamma^{cd}\pounds_n\gamma_{cd}\right) = -\frac{1}{2}\psi^4\left(\pounds_n\tilde{\gamma}_{ab} - \frac{1}{3}\tilde{\gamma}_{ab}\tilde{\gamma}^{cd}\pounds_n\tilde{\gamma}_{cd}\right) \tag{80}$$

$$= -\frac{\psi^4}{2\alpha}\left(\pounds_{\alpha n}\tilde{\gamma}_{ab} - \frac{1}{3}\tilde{\gamma}_{ab}\tilde{\gamma}^{cd}\pounds_{\alpha n}\tilde{\gamma}_{cd}\right) = -\frac{\psi^4}{2\alpha}\left(\pounds_{k-\omega}\tilde{\gamma}_{ab} - \frac{1}{3}\tilde{\gamma}_{ab}\tilde{\gamma}^{cd}\pounds_{k-\omega}\tilde{\gamma}_{cd}\right) \tag{81}$$

$$= \frac{\psi^4}{2\alpha}\left(\pounds_\omega\tilde{\gamma}_{ab} - \frac{1}{3}\tilde{\gamma}_{ab}\tilde{\gamma}^{cd}\pounds_\omega\tilde{\gamma}_{cd}\right) \tag{82}$$

$A_{ab} = \psi^4\tilde{A}_{ab}$, $(A_a{}^b = \tilde{A}_a{}^b)$

$$\tilde{A}_{ab} = \frac{1}{2\alpha}\left(\pounds_\omega\tilde{\gamma}_{ab} - \frac{1}{3}\tilde{\gamma}_{ab}\tilde{\gamma}^{cd}\pounds_\omega\tilde{\gamma}_{cd}\right) = \frac{1}{2\alpha}\left(\tilde{D}_a\tilde{\omega}_b + \tilde{D}_b\tilde{\omega}_a - \frac{2}{3}\tilde{\gamma}_{ab}\tilde{D}_c\tilde{\omega}^c\right) \tag{83}$$

F. Conformally related quatities

Ricci tensor

$$
{}^3R_{ab} = {}^3\tilde{R}_{ab} + {}^3\tilde{R}^{\psi}_{ab}, \tag{84}
$$

where

$$
{}^3\tilde{R}^{\psi}_{ab} = -\frac{2}{\psi}\tilde{D}_a\tilde{D}_b\psi - \tilde{\gamma}_{ab}\frac{2}{\psi}\tilde{D}^c\tilde{D}_c\psi + \frac{6}{\psi^2}\tilde{D}_a\psi\tilde{D}_b\psi - \tilde{\gamma}_{ab}\frac{2}{\psi^2}\tilde{D}_c\psi\tilde{D}^c\psi\,. \tag{85}
$$

Separate the elliptic operator for h_{ab} ${}^3\tilde{R}_{ab}$,

$$
{}^3\tilde{R}_{ab} = -\frac{1}{2}\mathring{D}^c\mathring{D}_c h_{ab} - \frac{1}{2}(f_{ac}\mathring{D}_b F^c + f_{bc}\mathring{D}_a F^c) + R^{\rm NL}_{ab}, \tag{86}
$$

where

$$
\begin{aligned}
R^{\rm NL}_{ab} \;=\; & -\frac{1}{2}(\mathring{D}_b h^{cd}\mathring{D}_c h_{ad} + \mathring{D}_a h^{cd}\mathring{D}_c h_{bd} + h^{cd}\mathring{D}_c\mathring{D}_d h_{ab}) - \mathring{D}_a C^c_{cb} + C^c_{ab}C^d_{dc} - C^d_{ac}C^c_{bd} \\
& -\frac{1}{2}[\,\mathring{D}_b(h_{ac}F^c) + \mathring{D}_a(h_{bc}F^c)\,] + F^c C_{c,ab}\,.
\end{aligned} \tag{87}
$$

Applying the generalized Dirac gauge condition and $\tilde{\gamma} = f$

$$
R^{\rm NL}_{ab} \;=\; -\frac{1}{2}(\mathring{D}_b h^{cd}\mathring{D}_c h_{ad} + \mathring{D}_a h^{cd}\mathring{D}_c h_{bd} + h^{cd}\mathring{D}_c\mathring{D}_d h_{ab}) - C^d_{ac}C^c_{bd} \tag{88}
$$

the first term in the r.h.s. of Eq. (86) becomes only linear part in both of h_{ab} and h^{ab}. A choice of Eq. (285) maintains rest of terms in Eq. (87) to be small to respect ellipticity of the equation.

Later, a combination of terms ${}^3R_{ab} - \dfrac{1}{\alpha}D_aD_b\alpha$ appears in the spatial projection of Einstein's equation. Confomal decomposition of the combination has the following form: two relations

$$
\begin{aligned}
\frac{1}{\alpha}D_aD_b\alpha \;&=\; \frac{1}{\alpha}(\tilde{D}_a\tilde{D}_b\alpha - \tilde{C}^c_{ab}\tilde{D}_c\alpha) && (89)\\
&=\; \frac{1}{\alpha}\tilde{D}_a\tilde{D}_b\alpha \;-\; \frac{2}{\alpha\psi}(\tilde{D}_a\alpha\tilde{D}_b\psi + \tilde{D}_b\alpha\tilde{D}_a\psi) \;+\; \tilde{\gamma}_{ab}\frac{2}{\alpha\psi}\tilde{D}^c\alpha\tilde{D}_c\psi, && (90)
\end{aligned}
$$

and

$$
\frac{1}{\alpha\psi^2}\tilde{D}_a\tilde{D}_b(\alpha\psi^2) = \frac{1}{\alpha}\tilde{D}_a\tilde{D}_b\alpha \;+\; \frac{2}{\psi}\tilde{D}_a\tilde{D}_b\psi \;+\; \frac{2}{\psi^2}\tilde{D}_a\psi\tilde{D}_b\psi \;+\; \frac{2}{\alpha\psi}(\tilde{D}_a\alpha\tilde{D}_b\psi + \tilde{D}_b\alpha\tilde{D}_a\psi), \tag{91}
$$

with Eqs.(84) and (85) yeilds,

$$
\begin{aligned}
{}^3R_{ab} - \frac{1}{\alpha}D_aD_b\alpha \;&=\; {}^3\tilde{R}_{ab} + {}^3\tilde{R}^{\psi}_{ab} - \frac{1}{\alpha}D_aD_b\alpha && (92)\\
&=\; {}^3\tilde{R}_{ab} \;-\; \frac{1}{\alpha\psi^2}\tilde{D}_a\tilde{D}_b(\alpha\psi^2) \;+\; \frac{4}{\alpha\psi^2}\left[\tilde{D}_a(\alpha\psi)\tilde{D}_b\psi + \tilde{D}_b(\alpha\psi)\tilde{D}_a\psi\right] \\
&\quad -\tilde{\gamma}_{ab}\left[\frac{2}{\psi}\tilde{D}^c\tilde{D}_c\psi + \frac{2}{\alpha\psi^2}\tilde{D}^c(\alpha\psi)\tilde{D}_c\psi\right] && (93)\\
&=\; {}^3\tilde{R}_{ab} \;-\; \frac{1}{\alpha\psi^2}\mathring{D}_a\mathring{D}_b(\alpha\psi^2) \;+\; \frac{1}{\alpha\psi^2}C^c_{ab}\mathring{D}_c(\alpha\psi^2) \\
&\qquad + \frac{4}{\alpha\psi^2}\left[\mathring{D}_a(\alpha\psi)\mathring{D}_b\psi + \mathring{D}_b(\alpha\psi)\mathring{D}_a\psi\right] \\
&\qquad - \tilde{\gamma}_{ab}\left[\frac{2}{\psi}\tilde{D}^c\tilde{D}_c\psi + \frac{2}{\alpha\psi^2}\tilde{D}^c(\alpha\psi)\tilde{D}_c\psi\right]. && (94)
\end{aligned}
$$

Finally the Ricci scalar curvature relates to the conformal Ricci scalar ${}^3R$ as

$$
{}^3R = {}^3\tilde{R}\,\psi^{-4} \;-\; 8\psi^{-5}\tilde{D}^a\tilde{D}_a\psi. \tag{95}
$$

III. EQUATIONS FOR THE FIELDS

A. Hamiltonian constraints

1. Full form

$$(G_{\alpha\beta} - 8\pi T_{\alpha\beta})n^\alpha n^\beta = 0 \tag{96}$$

$$(G_{\alpha\beta} - 8\pi T_{\alpha\beta})n^\alpha n^\beta = \frac{1}{2}({}^3R + K^2 - K_{ab}K^{ab} - 16\pi\rho_{\rm H}) \tag{97}$$

Since

$${}^3R = \frac{1}{\psi^4}{}^3\tilde{R} - \frac{8}{\psi^5}\tilde{D}^a\tilde{D}_a\psi. \tag{98}$$

$$(G_{\alpha\beta} - 8\pi T_{\alpha\beta})n^\alpha n^\beta = \frac{4}{\psi^5}\left[-\tilde{D}^a\tilde{D}_a\psi + \frac{\psi}{8}{}^3\tilde{R} - \frac{\psi^5}{8}\left(\tilde{A}_{ab}\tilde{A}^{ab} - \frac{2}{3}K^2\right) - 2\pi\psi^5\rho_{\rm H}\right] = 0. \tag{99}$$

B. Momentum constraints

1. Full form

$$(G_{\alpha\beta} - 8\pi T_{\alpha\beta})\gamma_a{}^\alpha n^\beta = 0 \tag{100}$$

$$\begin{aligned}
(G_{\alpha\beta} - 8\pi T_{\alpha\beta})\gamma_a{}^\alpha n^\beta = & \quad -D_b K_a{}^b + D_a K + 8\pi j_a & (101)\\
= & \quad -\frac{1}{\sqrt{\gamma}} D_b\left(\sqrt{\gamma} A_a{}^b\right) + \frac{2}{3} D_a K + 8\pi j_a & (102)
\end{aligned}$$

2. Reduction

Elliptic equations with respect to the shift are derived from the momentum constraint. It is solved for the covariant component of the non-rotating shift $\tilde\beta_a := \tilde\gamma_{ab}\beta^b$. Applying Eq.(53) to a tracefree A_{ab}, we have

$$\begin{aligned}
-\frac{1}{\sqrt{\gamma}} D_b\left(\sqrt{\gamma} A_a{}^b\right) + \frac{2}{3} D_a K + 8\pi j_a \;&=\; -\frac{1}{\sqrt{\gamma}} \tilde D_b\left(\sqrt{\gamma} \tilde A_a{}^b\right) \;+\; \frac{2}{3}\tilde D_a K \;+\; 8\pi j_a & (103)\\
&=\; -\frac{1}{\psi^6} \tilde D_b\left(\psi^6 \tilde A_a{}^b\right) \;+\; \frac{2}{3}\tilde D_a K \;+\; 8\pi j_a \;=\; 0 & (104)
\end{aligned}$$

As discussed in Sec.II E, imposing the helical symmetry $\pounds_k g_{\alpha\beta} = 0$, $\tilde A_{ab}$ is written,

$$\begin{aligned}
\tilde A_a{}^b \;&=\; \frac{1}{2\alpha}\left(\pounds_\omega \tilde\gamma_{ae} - \frac{1}{3}\tilde\gamma_{ae}\tilde\gamma^{cd}\pounds_\omega\tilde\gamma_{cd}\right)\tilde\gamma^{eb} & (105)\\
&=\; \frac{1}{2\alpha}\left(\tilde D_a\tilde\omega^b + \tilde D^b\tilde\omega_a - \frac{2}{3}\tilde\gamma_a{}^b\tilde D_c\tilde\omega^c\right) & (106)\\
&=\; \frac{1}{2\alpha}\left(\tilde D_a\tilde\beta^b + \tilde D^b\tilde\beta_a - \frac{2}{3}\tilde\gamma_a{}^b\tilde D_c\tilde\beta^c\right) + \frac{1}{2\alpha}\Omega(\tilde L\phi)_a{}^b, & (107)
\end{aligned}$$

where,

$$\begin{aligned}
(\tilde L\phi)_{ab} \;&=\; \pounds_\phi\tilde\gamma_{ab} - \frac{1}{3}\tilde\gamma_{ab}\tilde\gamma^{cd}\pounds_\phi\tilde\gamma_{cd} & (108)\\
&=\; \tilde D_a\tilde\phi_b + \tilde D_b\tilde\phi_a - \frac{2}{3}\tilde\gamma_{ab}\tilde D_c\tilde\phi^c. & (109)
\end{aligned}$$

In the above, $\tilde\phi^a = \phi^a$ and $\tilde\phi_a := \tilde\gamma_{ab}\tilde\phi^b$. Substituting Eq.(107) into (104), we have

$$\begin{aligned}
& \frac{1}{\psi^6}\tilde D_b\left[\psi^6\frac{1}{2\alpha}\left(\tilde D_a\tilde\beta^b + \tilde D^b\tilde\beta_a - \frac{2}{3}\tilde\gamma_a{}^b\tilde D_c\beta^c + \Omega(\tilde L\phi)_a{}^b\right)\right] \;-\; \frac{2}{3}\tilde D_a K \;=\; 8\pi j_a & (110)\\
\Leftrightarrow\quad & \frac{1}{2\alpha}\tilde D_b\left(\tilde D_a\tilde\beta^b + \tilde D^b\tilde\beta_a - \frac{2}{3}\tilde\gamma_a{}^b\tilde D_c\beta^c\right) \;+\; \frac{1}{2\alpha}\Omega\tilde D_b(\tilde L\phi)_a{}^b \;+\; \tilde A_a{}^b\frac{\alpha}{\psi^6}\tilde D_b\left(\frac{\psi^6}{\alpha}\right) \;-\; \frac{2}{3}\tilde D_a K \;=\; 8\pi j_a & (111)\\
\Leftrightarrow\quad & \tilde D_b\tilde D_a\tilde\beta^b + \tilde D_b\tilde D^b\tilde\beta_a - \frac{2}{3}\tilde D_a\tilde D_c\beta^c \;+\; \Omega\tilde D_b(\tilde L\phi)_a{}^b \;+\; 2\alpha\tilde A_a{}^b\frac{\alpha}{\psi^6}\tilde D_b\left(\frac{\psi^6}{\alpha}\right) \;-\; \frac{4}{3}\alpha\tilde D_a K \;=\; 16\pi\alpha j_a & (112)
\end{aligned}$$

Using

$$\tilde D_b\tilde D_a\beta^b - \tilde D_a\tilde D_b\beta^b = {}^3\tilde R_{ab}\beta^b, \tag{113}$$

we have

$$\tilde D_b\tilde D^b\tilde\beta_a + \frac{1}{3}\tilde D_a\tilde D_b\tilde\beta^b + {}^3\tilde R_{ab}\tilde\beta^b + \Omega\tilde D_b(\tilde L\phi)_a{}^b + 2\alpha\tilde A_a{}^b\frac{\alpha}{\psi^6}\tilde D_b\left(\frac{\psi^6}{\alpha}\right) - \frac{4}{3}\,\alpha\tilde D_a K \;=\; 16\pi\alpha j_a. \tag{114}$$

3. Shibata bunkai

From a principal part of Eq.(114), $\tilde{D}_b\tilde{D}^b\tilde{\beta}_a + \frac{1}{3}\tilde{D}_a\tilde{D}_b\tilde{\beta}^b$, the flat operator $\mathring{D}_b\mathring{D}^b\tilde{\beta}_a + \frac{1}{3}\mathring{D}_a\mathring{D}^b\tilde{\beta}_b$ is separated, then the Shibata decomposition,

$$\tilde{\beta}_a = G_a + \frac{1}{8}\mathring{D}_a(B - x^b G_b), \quad \text{where} \quad \mathring{D}_a x^b = \delta^b_a, \tag{115}$$

is applied. Note that $\mathring{D}^b\tilde{\beta}_b := f^{ab}\mathring{D}_a\tilde{\beta}_b$.

The separation goes as follows: the first term

$$\begin{aligned}
\tilde{D}_b\tilde{D}^b\tilde{\beta}_a = \tilde{\gamma}^{bc}\tilde{D}_b\tilde{D}_c\tilde{\beta}_a &= \tilde{\gamma}^{bc}\left[\mathring{D}_b(\tilde{D}_c\tilde{\beta}_a) - C^d_{bc}\tilde{D}_d\tilde{\beta}_a - C^d_{ba}\tilde{D}_c\tilde{\beta}_d\right] &(116)\\
&= \tilde{\gamma}^{bc}\left[\mathring{D}_b(\mathring{D}_c\tilde{\beta}_a - C^d_{ca}\tilde{\beta}_d) - C^d_{bc}\tilde{D}_d\tilde{\beta}_a - C^d_{ba}\tilde{D}_c\tilde{\beta}_d\right] &(117)\\
&= \mathring{\Delta}\tilde{\beta}_a + h^{bc}\mathring{D}_b\mathring{D}_c\tilde{\beta}_a - \tilde{\gamma}^{bc}\mathring{D}_b(C^d_{ca}\tilde{\beta}_d) - \tilde{\gamma}^{bc}C^d_{bc}\tilde{D}_d\tilde{\beta}_a - \tilde{\gamma}^{bc}C^d_{ba}\tilde{D}_c\tilde{\beta}_d, &(118)
\end{aligned}$$

and the second term

$$\begin{aligned}
\tilde{D}_a\tilde{D}_b\tilde{\beta}^b = \mathring{D}_a(\tilde{\gamma}^{bc}\tilde{D}_b\tilde{\beta}_c) &= \mathring{D}_a\left[f^{bc}(\mathring{D}_b\tilde{\beta}_c - C^d_{bc}\tilde{\beta}_d) + h^{bc}\tilde{D}_b\tilde{\beta}_c\right] &(119)\\
&= \mathring{D}_a\mathring{D}^b\tilde{\beta}_b + \mathring{D}_a(-f^{bc}C^d_{bc}\tilde{\beta}_d + h^{bc}\tilde{D}_b\tilde{\beta}_c) &(120)\\
&= \mathring{D}_a\left[\tilde{\gamma}^{bc}(\mathring{D}_b\tilde{\beta}_c - C^d_{bc}\tilde{\beta}_d)\right] &(121)\\
&= \mathring{D}_a\mathring{D}^b\tilde{\beta}_b + \mathring{D}_a(h^{bc}\mathring{D}_b\tilde{\beta}_c - \tilde{\gamma}^{bc}C^d_{bc}\tilde{\beta}_d), &(122)
\end{aligned}$$

are combined,

$$\begin{aligned}
\tilde{D}_b\tilde{D}^b\tilde{\beta}_a + \frac{1}{3}\tilde{D}_a\tilde{D}_b\tilde{\beta}^b &= \mathring{\Delta}\tilde{\beta}_a + \frac{1}{3}\mathring{D}_a\mathring{D}^b\tilde{\beta}_b + h^{bc}\mathring{D}_b\mathring{D}_c\tilde{\beta}_a - \tilde{\gamma}^{bc}\mathring{D}_b(C^d_{ca}\tilde{\beta}_d)\\
&\quad -\tilde{\gamma}^{bc}C^d_{bc}\tilde{D}_d\tilde{\beta}_a - \tilde{\gamma}^{bc}C^d_{ba}\tilde{D}_c\tilde{\beta}_d + \frac{1}{3}\mathring{D}_a(-f^{bc}C^d_{bc}\tilde{\beta}_d + h^{bc}\tilde{D}_b\tilde{\beta}_c) &(123)\\
&= \mathring{\Delta}\tilde{\beta}_a + \frac{1}{3}\mathring{D}_a\mathring{D}^b\tilde{\beta}_b + h^{bc}\mathring{D}_b\mathring{D}_c\tilde{\beta}_a - \tilde{\gamma}^{bc}\mathring{D}_b(C^d_{ca}\tilde{\beta}_d)\\
&\quad -\tilde{\gamma}^{bc}C^d_{bc}\tilde{D}_d\tilde{\beta}_a - \tilde{\gamma}^{bc}C^d_{ba}\tilde{D}_c\tilde{\beta}_d + \frac{1}{3}\mathring{D}_a(h^{bc}\mathring{D}_b\tilde{\beta}_c - \tilde{\gamma}^{bc}C^d_{bc}\tilde{\beta}_d). &(124)
\end{aligned}$$

A substitution $\tilde{\beta}_a = G_a + \frac{1}{8}\mathring{D}_a(B - x^b G_b)$ into the flat operator $\mathring{D}_b\mathring{D}^b\tilde{\beta}_a + \frac{1}{3}\mathring{D}_a\mathring{D}^b\tilde{\beta}_b$,

$$\begin{aligned}
\mathring{\Delta}\tilde{\beta}_a + \frac{1}{3}\mathring{D}_a\mathring{D}^b\tilde{\beta}_b &= \mathring{\Delta}G_a + \frac{1}{8}\mathring{D}_a\,\mathring{\Delta}(B - x^b G_b) + \frac{1}{3}\mathring{D}_a\mathring{D}^b\left[G_b + \frac{1}{8}\mathring{D}_b(B - x^c G_c)\right] &(125)\\
&= \mathring{\Delta}G_a + \frac{1}{6}\mathring{D}_a(\,\mathring{\Delta}B - x^b\,\mathring{\Delta}G_b), &(126)
\end{aligned}$$

yields elliptic equations to be solved separately,

$$\begin{aligned}
\mathring{\Delta}G_a &= -h^{bc}\mathring{D}_b\mathring{D}_c\tilde{\beta}_a + \tilde{\gamma}^{bc}\mathring{D}_b(C^d_{ca}\tilde{\beta}_d) + \tilde{\gamma}^{bc}C^d_{bc}\tilde{D}_d\tilde{\beta}_a + \tilde{\gamma}^{bc}C^d_{ba}\tilde{D}_c\tilde{\beta}_d - \frac{1}{3}\mathring{D}_a(h^{bc}\mathring{D}_b\tilde{\beta}_c - \tilde{\gamma}^{bc}C^d_{bc}\tilde{\beta}_d)\\
&\quad -{}^3\tilde{R}_{ab}\tilde{\beta}^b - \Omega\tilde{D}_b(\tilde{L}\phi)_a{}^b - 2\alpha\tilde{A}_a{}^b\frac{\alpha}{\psi^6}\tilde{D}_b\left(\frac{\psi^6}{\alpha}\right) + 16\pi\alpha j_a &(127)\\
&=: S_a,
\end{aligned}$$

and

$$\mathring{\Delta}B = x^a\,\mathring{\Delta}G_a = x^a S_a. \tag{128}$$

Since $C^b_{ba} = 0$, a term $\tilde{D}_b(\tilde{L}\phi)_a{}^b$ is computed from

$$\begin{aligned}
\tilde{D}_b(\tilde{L}\phi)_a{}^b &= \mathring{D}_b(\tilde{L}\phi)_a{}^b - C^c_{ba}(\tilde{L}\phi)_c{}^b\\
&= \tilde{\gamma}^{bc}\mathring{D}_c(\tilde{L}\phi)_{ab} + (\tilde{L}\phi)_{ab}\mathring{D}_c\tilde{\gamma}^{bc} - C^c_{ba}(\tilde{L}\phi)_c{}^b\\
&= \tilde{\gamma}^{bc}\mathring{D}_c(\tilde{L}\phi)_{ab} - C^c_{ba}(\tilde{L}\phi)_{cd}\tilde{\gamma}^{bd}. &(129)
\end{aligned}$$

(For the comparison with the shift equation in the waveless spacetime, see Sec.III E.)

C. Spatial trace part of the Einstein equations

1. Equation for $\alpha\psi^2$

$$(G_{\alpha\beta} - 8\pi T_{\alpha\beta})\gamma^{\alpha\beta} = 0 \tag{130}$$

$$(G_{\alpha\beta} - 8\pi T_{\alpha\beta})\gamma^{\alpha\beta} = 2\pounds_n K \,-\, \frac{1}{2}({}^3R + K^2 + 3K_{ab}K^{ab}) \,+\, \frac{2}{\alpha}D^a D_a \alpha \,-\, 8\pi S = 0. \tag{131}$$

Since

$$-\frac{1}{2}{}^3R \,+\, \frac{2}{\alpha}D^a D_a\alpha \;=\; -\frac{1}{2}\gamma^{ab}({}^3\tilde{R}_{ab} + {}^3\tilde{R}^{\psi}_{ab}) \,+\, \frac{2}{\alpha}D^a D_a\alpha \tag{132}$$

$$= \; -\frac{1}{2\psi^4}{}^3\tilde{R} \,+\, \frac{2}{\alpha\psi^6}\left[\tilde{D}^a\tilde{D}_a(\alpha\psi^2) - 2\tilde{D}_a(\alpha\psi)\tilde{D}^a\psi\right], \tag{133}$$

the trace part becomes

$$(G_{\alpha\beta} - 8\pi T_{\alpha\beta})\gamma^{\alpha\beta}$$

$$= \frac{2}{\alpha\psi^6}\left[\tilde{D}^a\tilde{D}_a(\alpha\psi^2) \,-\, 2\tilde{D}_a(\alpha\psi)\tilde{D}^a\psi \,-\, \frac{\alpha\psi^2}{4}{}^3\tilde{R} \,+\, \alpha\psi^6\pounds_n K \,-\, \frac{\alpha\psi^6}{4}(K^2 + 3K_{ab}K^{ab}) \,-\, 4\pi\alpha\psi^6 S\right] \tag{134}$$

$$= \frac{2}{\alpha\psi^6}\left[\tilde{D}^a\tilde{D}_a(\alpha\psi^2) \,-\, 2\tilde{D}_a(\alpha\psi)\tilde{D}^a\psi \,-\, \frac{\alpha\psi^2}{4}{}^3\tilde{R} \,-\, \psi^6\pounds_\omega K \,-\, \frac{\alpha\psi^6}{4}(3\tilde{A}_{ab}\tilde{A}^{ab} + 2K^2) \,-\, 4\pi\alpha\psi^6 S\right] = 0. \tag{135}$$

2. Equation for $\alpha\psi$

Combine with the Hamiltonian constraint

$$(G_{\alpha\beta} - 8\pi T_{\alpha\beta})(\gamma^{\alpha\beta} + \frac{1}{2}n^\alpha n^\beta) = 0, \tag{136}$$

$$(G_{\alpha\beta} - 8\pi T_{\alpha\beta})(\gamma^{\alpha\beta} + \frac{1}{2}n^\alpha n^\beta) \;=\; -\frac{1}{4}{}^3R \,+\, \frac{2}{\alpha}D^a D_a\alpha \,+\, 2\pounds_n K \,-\, \frac{1}{4}(K^2 + 7K_{ab}K^{ab}) \,-\, 4\pi(\rho_{\rm H} + 2S) = 0. \tag{137}$$

Since

$$-\frac{1}{4}{}^3R \,+\, \frac{2}{\alpha}D^a D_a\alpha \;=\; -\frac{1}{4}\left(\frac{1}{\psi^4}{}^3\tilde{R} \,-\, \frac{8}{\psi^5}\tilde{D}^a\tilde{D}_a\psi\right) \,+\, \frac{2}{\alpha\psi^5}\left[\tilde{D}^a\tilde{D}_a(\alpha\psi) - \alpha\tilde{D}^a\tilde{D}_a\psi\right] \tag{138}$$

$$= \; \frac{2}{\alpha\psi^5}\left[\tilde{D}^a\tilde{D}_a(\alpha\psi) \,-\, \frac{\alpha\psi}{8}\,{}^3\tilde{R}\right] \tag{139}$$

$$(G_{\alpha\beta} - 8\pi T_{\alpha\beta})(\gamma^{\alpha\beta} + \frac{1}{2}n^\alpha n^\beta)$$

$$= \; \frac{2}{\alpha\psi^5}\left[\tilde{D}^a\tilde{D}_a(\alpha\psi) \,-\, \frac{\alpha\psi}{8}\,{}^3\tilde{R} \,+\, \alpha\psi^5\pounds_n K \,-\, \frac{1}{8}\alpha\psi^5\left(K^2 + 7K_{ab}K^{ab}\right) \,-\, 2\pi\,\alpha\psi^5(\rho_{\rm H} + 2S)\right] \tag{140}$$

$$= \; \frac{2}{\alpha\psi^5}\left[\tilde{D}^a\tilde{D}_a(\alpha\psi) \,-\, \frac{\alpha\psi}{8}\,{}^3\tilde{R} \,-\, \psi^5\pounds_\omega K \,-\, \alpha\psi^5\left(\frac{7}{8}\tilde{A}_{ab}\tilde{A}^{ab} + \frac{5}{12}K^2\right) \,-\, 2\pi\,\alpha\psi^5(\rho_{\rm H} + 2S)\right] = 0. \tag{141}$$

D. Spatial trace free part of the Einstein equations

1. Full form

$$(G_{\alpha\beta}-8\pi T_{\alpha\beta})(\gamma_a{}^\alpha\gamma_b{}^\beta-\frac{1}{3}\gamma_{ab}\gamma^{\alpha\beta})=0 \tag{142}$$

$$\begin{aligned}(G_{\alpha\beta}-8\pi T_{\alpha\beta})\gamma_a{}^\alpha\gamma_b{}^\beta = &\quad -£_n K_{ab}+\gamma_{ab}£_n K+{}^3R_{ab}-\frac{1}{2}\gamma_{ab}\,{}^3R+KK_{ab}-2K_{ac}K_b{}^c\\ &\quad -\frac{1}{2}\gamma_{ab}\left(K^2+K_{cd}K^{cd}\right)-\frac{1}{\alpha}(D_aD_b\alpha-\gamma_{ab}D^cD_c\alpha)-8\pi S_{ab}\\ = &\quad -£_n K_{ab}+{}^3R_{ab}+KK_{ab}-2K_{ac}K_b{}^c-\frac{1}{\alpha}D_aD_b\alpha-8\pi S_{ab}\\ &\quad +\gamma_{ab}\left[£_nK-\frac{1}{2}{}^3R-\frac{1}{2}\left(K^2+K_{cd}K^{cd}\right)+\frac{1}{\alpha}D^cD_c\alpha\right]\end{aligned} \tag{143}$$

$$\mathcal{E}_{ab}:=-£_nK_{ab}+{}^3R_{ab}+KK_{ab}-2K_{ac}K_b{}^c-\frac{1}{\alpha}D_aD_b\alpha-8\pi S_{ab} \tag{144}$$

$$(G_{\alpha\beta}-8\pi T_{\alpha\beta})(\gamma_a{}^\alpha\gamma_b{}^\beta-\frac{1}{3}\gamma_{ab}\gamma^{\alpha\beta})=\mathcal{E}_{ab}-\frac{1}{3}\gamma_{ab}\gamma^{cd}\mathcal{E}_{cd}=\mathcal{E}_{ab}-\frac{1}{3}\tilde\gamma_{ab}\tilde\gamma^{cd}\mathcal{E}_{cd}=0 \tag{145}$$

(**N.B.** As in Shibata and Uryu, one can substitute fluid source in the terms proportional to γ_{ab} in Eq.(143)) then could solve the projection without separating a tracefree part.)
(**N.B.** Show that the trace free part can be written

$$£_nA_a{}^b\;=\;KA_a{}^b+(\gamma_a{}^c\gamma_d{}^b-\frac{1}{3}\gamma_a{}^b\gamma_d{}^c)\left(R_c{}^d-\frac{1}{\alpha}D_cD^d\alpha-8\pi S_c{}^d\right) \tag{146}$$

2. Helmholtz equation

The above Eq.(145) is rewrittten in the form

$$\Box h_{ab}=\mathcal{S}_{ab}, \tag{147}$$

where $\Box$ is the flat d'Alembertian operator. With a helical symmetry,

$$\Box h_{ab}=-\partial_t^2h_{ab}+\mathring{D}^c\mathring{D}_ch_{ab}=(\mathring{\Delta}-£_{\Omega\phi}^2)h_{ab}=(\mathring{\Delta}-\Omega^2£_\phi^2)h_{ab}, \tag{148}$$

where $\mathring{D}^c=f^{cd}\mathring{D}_d$, $\mathring{\Delta}=\mathring{D}^c\mathring{D}_c$, $£_{\Omega\phi}^2=£_{\Omega\phi}£_{\Omega\phi}$ and $£_{\Omega\phi}=\Omega£_\phi$, since $\Omega=$ constant. Note that $£_\phi f_{ab}=0$ imply

$$£_\phi^2h_{ab}=£_\phi^2\tilde\gamma_{ab}. \tag{149}$$

Also

$$\mathring{\Delta}h_{ab}=\mathring{\Delta}\tilde\gamma_{ab}. \tag{150}$$

Separation of $\partial_\phi^2\tilde\gamma_{ab}$ from $£_\phi^2\tilde\gamma_{ab}$, where $\partial_\phi:=\phi^a\mathring{D}_a$ goes as,

$$\begin{aligned}£_\phi£_\phi\tilde\gamma_{ab}\;&=\;\phi^c\mathring{D}_c£_\phi\tilde\gamma_{ab}\;+\;£_\phi\tilde\gamma_{ac}\mathring{D}_b\phi^c\;+\;£_\phi\tilde\gamma_{cb}\mathring{D}_a\phi^c && (151)\\ &=\;\phi^c\mathring{D}_c\left(\phi^d\mathring{D}_d\tilde\gamma_{ab}+\tilde\gamma_{ad}\mathring{D}_b\phi^d+\tilde\gamma_{db}\mathring{D}_a\phi^d\right)\;+\;£_\phi\tilde\gamma_{ac}\mathring{D}_b\phi^c\;+\;£_\phi\tilde\gamma_{cb}\mathring{D}_a\phi^c && (152)\\ &=\;\partial_\phi^2\tilde\gamma_{ab}\;+\;(\partial_\phi\tilde\gamma_{ac}+£_\phi\tilde\gamma_{ac})\,\mathring{D}_b\phi^c\;+\;(\partial_\phi\tilde\gamma_{cb}+£_\phi\tilde\gamma_{cb})\,\mathring{D}_a\phi^c && (153)\end{aligned}$$

where a relation $\phi^c \mathring{D}_c(\mathring{D}_a\phi^b)=0$ is used. In Cartesian, $\mathring{D}_a\phi^b$ has components,

$$\mathring{D}_a\phi^b = \hat{x}_a\hat{y}^b - \hat{y}_a\hat{x}^b = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{where Cartesian basis are} \quad \hat{x}^a = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \hat{y}^a = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ \hat{z}^a = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{154}$$

Transposing extra terms to R.H.S., we have we separte the Helmholts operater in the L.H.S.,

$$(\mathring{\Delta} - \Omega^2\partial_\phi^2)\tilde{\gamma}_{ab} = \bar{\mathcal{S}}_{ab}, \tag{155}$$

or

$$(\mathring{\Delta} - \Omega^2\partial_\phi^2)h_{ab} = \bar{\mathcal{S}}_{ab}, \tag{156}$$

where $\partial_\phi f_{ab} = 0$.

3. Separating $\Omega^2\partial_\phi^2 h_{ab}$ term

A term $\pounds_{\Omega\phi}^2\tilde{\gamma}_{ab}$ appears from $-\pounds_n K_{ab}$ term in Eq.(144).

$$\begin{aligned} K_{ab} &= \psi^4\tilde{K}_{ab} = -\frac{1}{2}\pounds_n\gamma_{ab} = -\frac{1}{2\alpha}\pounds_{\alpha n}\gamma_{ab} && (157)\\ &= -\frac{\psi^4}{2\alpha}\left(\pounds_{\alpha n}\tilde{\gamma}_{ab} + \tilde{\gamma}_{ab}\pounds_{\alpha n}\ln\psi^4\right). && (158)\end{aligned}$$

$$\alpha\psi^{-4}K_{ab} = \alpha\tilde{K}_{ab} = \frac{1}{2}\pounds_\omega\tilde{\gamma}_{ab} + \frac{1}{2}\tilde{\gamma}_{ab}\pounds_\omega\ln\psi^4. \tag{159}$$

where $\pounds_{\alpha n}\tilde{\gamma}_{cd} = \pounds_{k-\omega}\tilde{\gamma}_{cd}$, $\pounds_k\tilde{\gamma}_{cd}=0$ and $\pounds_\omega = \pounds_\beta + \pounds_{\Omega\phi}$ are used.

$$\begin{aligned} -\pounds_n K_{ab} &= -\frac{1}{\alpha}\pounds_{\alpha n}K_{ab} = \frac{1}{\alpha}\pounds_\omega K_{ab} = \frac{\psi^4}{\alpha^2}\pounds_\omega\left(\alpha\psi^{-4}K_{ab}\right) - \frac{\psi^4}{\alpha^2}K_{ab}\pounds_\omega\left(\alpha\psi^{-4}\right) && (160)\\ &= \frac{\psi^4}{2\alpha^2}\pounds_\omega\pounds_\omega\tilde{\gamma}_{ab} + \frac{\psi^4}{2\alpha^2}\pounds_\omega\left(\tilde{\gamma}_{ab}\pounds_\omega\ln\psi^4\right) + \frac{1}{\alpha}K_{ab}\pounds_\omega\ln\frac{\psi^4}{\alpha} && (161)\\ &= \frac{\psi^4}{2\alpha^2}\pounds_\omega\pounds_\omega\tilde{\gamma}_{ab} + \frac{1}{\alpha}K_{ab}\pounds_\omega\ln\frac{\psi^8}{\alpha} + \frac{\psi^4}{2\alpha^2}\tilde{\gamma}_{ab}\left(\pounds_\omega\pounds_\omega\ln\psi^4 - \pounds_\omega\ln\psi^4\pounds_\omega\ln\psi^4\right). && (162)\end{aligned}$$

The first term of the R.H.S. becomes

$$\begin{aligned} \frac{\psi^4}{2\alpha^2}\pounds_\omega\pounds_\omega\tilde{\gamma}_{ab} &= \frac{\psi^4}{2\alpha^2}\pounds_{\beta+\Omega\phi}\pounds_{\beta+\Omega\phi}\tilde{\gamma}_{ab} && (163)\\ &= \frac{\psi^4}{2\alpha^2}\Omega^2\pounds_\phi\pounds_\phi\tilde{\gamma}_{ab} + \frac{\psi^4}{2\alpha^2}\Omega\left(\pounds_\phi\pounds_\beta\tilde{\gamma}_{ab} + \pounds_\beta\pounds_\phi\tilde{\gamma}_{ab}\right) + \frac{\psi^4}{2\alpha^2}\pounds_\beta\pounds_\beta\tilde{\gamma}_{ab}. && (164)\end{aligned}$$

Note that, for a scalar F,

$$\pounds_\omega F = \omega^a\mathring{D}_a F. \tag{165}$$

Also note that for a flat metric,

$$\pounds_\phi f_{ab} = 0, \quad \text{and hence} \quad \pounds_\phi\tilde{\gamma}_{ab} = \pounds_\phi h_{ab}. \tag{166}$$

Using Eq.(164), a term $\Omega^2\pounds_\phi^2 h_{ab}$, is separated from $-\pounds_n K_{ab}$ term:

$$\begin{aligned} -\pounds_n K_{ab} = {} & \frac{\psi^4}{2\alpha^2}\Omega^2\pounds_\phi^2 h_{ab} + \frac{\psi^4}{2\alpha^2}\Omega\left(\pounds_\phi\pounds_\beta\tilde{\gamma}_{ab} + \pounds_\beta\pounds_\phi\tilde{\gamma}_{ab}\right) + \frac{\psi^4}{2\alpha^2}\pounds_\beta\pounds_\beta\tilde{\gamma}_{ab} \\ & + \frac{1}{\alpha}K_{ab}\pounds_\omega\ln\frac{\psi^8}{\alpha} + \frac{\psi^4}{2\alpha^2}\tilde{\gamma}_{ab}\left(\pounds_\omega\pounds_\omega\ln\psi^4 - \pounds_\omega\ln\psi^4\pounds_\omega\ln\psi^4\right). \end{aligned} \tag{167}$$

With a help of Eq.(153), we separate $\Omega^2\partial_\phi^2 h_{ab}$ as

$$\begin{aligned}-\pounds_n K_{ab} \;=\;& \frac{\psi^4}{2\alpha^2}\,\Omega^2\partial_\phi^2 h_{ab} \;+\; \frac{\psi^4}{2\alpha^2}\,\Omega^2\left[\left(\partial_\phi\tilde\gamma_{ac}+\pounds_\phi\tilde\gamma_{ac}\right)\overset{\circ}{D}_b\phi^c \;+\; \left(\partial_\phi\tilde\gamma_{cb}+\pounds_\phi\tilde\gamma_{cb}\right)\overset{\circ}{D}_a\phi^c\right] \\ &+\frac{\psi^4}{2\alpha^2}\,\Omega\left(\pounds_\phi\pounds_\beta\tilde\gamma_{ab}+\pounds_\beta\pounds_\phi\tilde\gamma_{ab}\right) \;+\; \frac{\psi^4}{2\alpha^2}\,\pounds_\beta\pounds_\beta\tilde\gamma_{ab} \;+\; \frac{1}{\alpha}K_{ab}\,\pounds_\omega \ln\frac{\psi^8}{\alpha} \\ &+\frac{\psi^4}{2\alpha^2}\tilde\gamma_{ab}\,\left(\pounds_\omega\pounds_\omega\ln\psi^4-\pounds_\omega\ln\psi^4\,\pounds_\omega\ln\psi^4\right). \end{aligned} \tag{168}$$

Imposing the maximal slicing $K=0$, we can replace

$$K_{ab}\to A_{ab}=\psi^4\tilde A_{ab}, \qquad \tilde K_{ab}\to\tilde A_{ab}, \tag{169}$$

Also with the helical symmetry we have

$$-\pounds_n K_{ab}=\frac{1}{\alpha}\pounds_\omega(\psi^4\tilde A_{ab}). \tag{170}$$

4. Separating $\overset{\circ}{\Delta} h_{ab}$ term

A term $\overset{\circ}{\Delta} h_{ab}$ is found in ${}^3R_{ab}$ as having been done in Sec.II F. In the Dirac gauge, it is written

$${}^3\tilde R_{ab}=-\frac{1}{2}\overset{\circ}{D}{}^c\overset{\circ}{D}_c h_{ab}+R^{\rm NL}_{ab}. \tag{171}$$

5. Separating Helmholtz operator

Eq.(144) with the helical symmetry is rewritten,

$$\mathcal{E}_{ab}:=\frac{1}{\alpha}\pounds_\omega K_{ab}+{}^3R_{ab}-\frac{1}{\alpha}D_aD_b\alpha+KK_{ab}-2K_{ac}K_b{}^c-8\pi S_{ab} \tag{172}$$

Substituting Eqs.(168) and (171),

$$\begin{aligned}\mathcal{E}_{ab} \;=\;& -\frac{1}{2}\left(\overset{\circ}{D}{}^c\overset{\circ}{D}_c\;-\;\Omega^2\partial_\phi^2\right)h_{ab}+R^{\rm NL}_{ab}+{}^3\tilde R^\psi_{ab}-\frac{1}{\alpha}D_aD_b\alpha+KK_{ab}-2K_{ac}K_b{}^c-8\pi S_{ab} \\ &+\frac{1}{2}\left(\frac{\psi^4}{\alpha^2}-1\right)\,\Omega^2\partial_\phi^2 h_{ab} \;+\; \frac{\psi^4}{2\alpha^2}\,\Omega^2\left[\left(\partial_\phi\tilde\gamma_{ac}+\pounds_\phi\tilde\gamma_{ac}\right)\overset{\circ}{D}_b\phi^c \;+\; \left(\partial_\phi\tilde\gamma_{cb}+\pounds_\phi\tilde\gamma_{cb}\right)\overset{\circ}{D}_a\phi^c\right] \\ &+\frac{\psi^4}{2\alpha^2}\,\Omega\left(\pounds_\phi\pounds_\beta\tilde\gamma_{ab}+\pounds_\beta\pounds_\phi\tilde\gamma_{ab}\right) \;+\; \frac{\psi^4}{2\alpha^2}\,\pounds_\beta\pounds_\beta\tilde\gamma_{ab} \;+\; \frac{1}{\alpha}K_{ab}\,\pounds_\omega\ln\frac{\psi^8}{\alpha} \\ &+\frac{\psi^4}{2\alpha^2}\tilde\gamma_{ab}\,\left(\pounds_\omega\pounds_\omega\ln\psi^4-\pounds_\omega\ln\psi^4\,\pounds_\omega\ln\psi^4\right). \end{aligned} \tag{173}$$

Alternatively, one can simply write

$$\begin{aligned}\mathcal{E}_{ab} \;=\;& -\frac{1}{2}\left(\overset{\circ}{D}{}^c\overset{\circ}{D}_c\;-\;\Omega^2\partial_\phi^2\right)h_{ab}+R^{\rm NL}_{ab}+{}^3\tilde R^\psi_{ab}-\frac{1}{\alpha}D_aD_b\alpha+KK_{ab}-2K_{ac}K_b{}^c-8\pi S_{ab} \\ &+\frac{1}{\alpha}\pounds_\omega K_{ab}-\frac{1}{2}\Omega^2\partial_\phi^2 h_{ab} \end{aligned} \tag{174}$$

6. Separating Helmholtz operator in $K=0$ slice

Eq.(144) with the helical symmetry, and $\tilde A_{ab}$ instead of K_{ab}, is rewritten,

$$\mathcal{E}_{ab}:=\frac{1}{\alpha}\pounds_\omega(\psi^4\tilde A_{ab})+{}^3\tilde R_{ab}+{}^3\tilde R^\psi_{ab}-2\psi^4\tilde A_{ac}\tilde A_b{}^c-\frac{1}{\alpha}D_aD_b\alpha-8\pi S_{ab}. \tag{175}$$

Substituting Eqs.(168) and (171),

$$\begin{aligned}\mathcal{E}_{ab} \;=\; & -\frac{1}{2}\left(\mathring{D}^c\mathring{D}_c \,-\, \Omega^2\partial_\phi^2\right) h_{ab} + R^{\rm NL}_{ab} + {}^3\tilde{R}^{\psi}_{ab} - 2\psi^4\tilde{A}_{ac}\tilde{A}_b{}^c - \frac{1}{\alpha}D_aD_b\alpha - 8\pi S_{ab} \\ & +\frac{1}{2}\left(\frac{\psi^4}{\alpha^2}-1\right)\,\Omega^2\partial_\phi^2 h_{ab} \;+\; \frac{\psi^4}{2\alpha^2}\,\Omega^2\left[(\partial_\phi\tilde{\gamma}_{ac}+\pounds_\phi\tilde{\gamma}_{ac})\,\mathring{D}_b\phi^c \;+\; (\partial_\phi\tilde{\gamma}_{cb}+\pounds_\phi\tilde{\gamma}_{cb})\,\mathring{D}_a\phi^c\right] \\ & +\frac{\psi^4}{2\alpha^2}\,\Omega\,(\pounds_\phi\pounds_\beta\tilde{\gamma}_{ab}+\pounds_\beta\pounds_\phi\tilde{\gamma}_{ab}) \;+\; \frac{\psi^4}{2\alpha^2}\,\pounds_\beta\pounds_\beta\tilde{\gamma}_{ab} \;+\; \frac{\psi^4}{\alpha}\tilde{A}_{ab}\,\pounds_\omega\ln\frac{\psi^8}{\alpha} \\ & +\frac{\psi^4}{2\alpha^2}\tilde{\gamma}_{ab}\,\left(\pounds_\omega\pounds_\omega\ln\psi^4 - \pounds_\omega\ln\psi^4\,\pounds_\omega\ln\psi^4\right). \end{aligned} \tag{176}$$

Alternatively, one can simply write

$$\begin{aligned}\mathcal{E}_{ab} \;=\; & -\frac{1}{2}\left(\mathring{D}^c\mathring{D}_c \,-\, \Omega^2\partial_\phi^2\right) h_{ab} + R^{\rm NL}_{ab} + {}^3\tilde{R}^{\psi}_{ab} - 2\psi^4\tilde{A}_{ac}\tilde{A}_b{}^c - \frac{1}{\alpha}D_aD_b\alpha - 8\pi S_{ab} \\ & +\frac{1}{\alpha}\pounds_\omega(\psi^4\tilde{A}_{ab}) - \frac{1}{2}\Omega^2\partial_\phi^2 h_{ab}\end{aligned} \tag{177}$$

7. Tracefree operation

One can drop terms propotional to $\tilde{\gamma}_{ab}$, which will not remain after the trace free operation, $\tilde{\gamma}_a{}^c\tilde{\gamma}_b{}^d - \frac{1}{3}\tilde{\gamma}_{ab}\tilde{\gamma}^{cd}$.

$$ {}^3\bar{R}^{\psi}_{ab} = {}^3\tilde{R}^{\psi}_{ab} - \frac{1}{3}\tilde{\gamma}_{ab}\tilde{\gamma}^{cd}\,{}^3\tilde{R}^{\psi}_{ab} = -\frac{2}{\psi}\tilde{D}_a\tilde{D}_b\psi + \frac{6}{\psi^2}\tilde{D}_a\psi\tilde{D}_b\psi \tag{178}$$

$$ D_aD_b\alpha \;=\; \tilde{D}_a\tilde{D}_b\alpha - \tilde{C}^c_{ab}\tilde{D}_c\alpha \;=\; \mathring{D}_a\mathring{D}_b\alpha - C^c_{ab}\mathring{D}_c\alpha - \tilde{C}^c_{ab}\mathring{D}_c\alpha \tag{179}$$

$$ \tilde{C}^c_{ab}\mathring{D}_c\alpha = \frac{2}{\psi}(\tilde{\gamma}^c_a\mathring{D}_b\psi + \tilde{\gamma}^c_b\mathring{D}_a\psi - \tilde{\gamma}_{ab}\tilde{\gamma}^{cd}\mathring{D}_d\psi)\mathring{D}_c\alpha \tag{180}$$

$$ \bar{D}_a\bar{D}_b\alpha \;:=\; \left[\mathring{D}_a\mathring{D}_b\alpha - C^c_{ab}\mathring{D}_c\alpha - \frac{2}{\psi}(\mathring{D}_a\alpha\mathring{D}_b\psi + \mathring{D}_b\alpha\mathring{D}_a\psi)\right] \tag{181}$$

With a help of Eq.(94),

$$ {}^3\bar{R}^{\psi}_{ab} - \frac{1}{\alpha}\bar{D}_a\bar{D}_b\alpha \;=\; -\,\frac{1}{\alpha\psi^2}\mathring{D}_a\mathring{D}_b(\alpha\psi^2) \;+\; \frac{1}{\alpha\psi^2}C^c_{ab}\mathring{D}_c(\alpha\psi^2) \;+\; \frac{4}{\alpha\psi^2}\left[\mathring{D}_a(\alpha\psi)\mathring{D}_b\psi + \mathring{D}_b(\alpha\psi)\mathring{D}_a\psi\right], \tag{182}$$

where note

$$ \left(\tilde{\gamma}_a{}^c\tilde{\gamma}_b{}^d - \frac{1}{3}\tilde{\gamma}_{ab}\tilde{\gamma}^{cd}\right)\left({}^3\tilde{R}^{\psi}_{cd} - \frac{1}{\alpha}D_cD_d\alpha\right) \;=\; \left(\tilde{\gamma}_a{}^c\tilde{\gamma}_b{}^d - \frac{1}{3}\tilde{\gamma}_{ab}\tilde{\gamma}^{cd}\right)\left({}^3\bar{R}^{\psi}_{cd} - \frac{1}{\alpha}\bar{D}_c\bar{D}_d\alpha\right) \tag{183}$$

For the matter source term,

$$ S_{ab} = \frac{\rho}{h}\mathring{D}_a\Phi\,\mathring{D}_b\Phi + \gamma_{ab}p \tag{184}$$

$$ \bar{S}_{ab} = \frac{\rho}{h}\mathring{D}_a\Phi\,\mathring{D}_b\Phi \tag{185}$$

Trace free operation for the Helmholtz operator becomes

$$\begin{aligned} & -\frac{1}{2}\left(\tilde{\gamma}_a{}^c\tilde{\gamma}_b{}^d - \frac{1}{3}\tilde{\gamma}_{ab}\tilde{\gamma}^{cd}\right)\left(\mathring{D}^c\mathring{D}_c \,-\, \Omega^2\partial_\phi^2\right) h_{cd} \\ & \quad = -\frac{1}{2}\left[\left(\mathring{D}^c\mathring{D}_c \,-\, \Omega^2\partial_\phi^2\right) h_{ab} - \frac{1}{3}\tilde{\gamma}_{ab}\tilde{\gamma}^{cd}\left(\mathring{D}^e\mathring{D}_e \,-\, \Omega^2\partial_\phi^2\right) h_{cd}\right]\end{aligned} \tag{186}$$

For the last term,

$$
\begin{aligned}
-\tilde{\gamma}^{cd}\left(\mathring{D}^e\mathring{D}_e - \Omega^2\partial_\phi^2\right)h_{cd} &= -\tilde{\gamma}^{cd}\left(\mathring{D}^e\mathring{D}_e - \Omega^2\partial_\phi^2\right)\tilde{\gamma}_{cd} \\
&= \mathring{D}^e\tilde{\gamma}^{cd}\,\mathring{D}_e\tilde{\gamma}_{cd} - \Omega^2\partial_\phi\tilde{\gamma}^{cd}\partial_\phi\tilde{\gamma}_{cd} \\
&\quad -\left[\mathring{D}^e\left(\tilde{\gamma}^{cd}\mathring{D}_e\tilde{\gamma}_{cd}\right) - \Omega^2\partial_\phi\left(\tilde{\gamma}^{cd}\partial_\phi\tilde{\gamma}_{cd}\right)\right] \\
&= \mathring{D}^e h^{cd}\,\mathring{D}_e h_{cd} - \Omega^2\partial_\phi h^{cd}\partial_\phi h_{cd}.
\end{aligned} \tag{187}
$$

Therefore,

$$
\begin{aligned}
&-\frac{1}{2}\left(\tilde{\gamma}_a{}^c\tilde{\gamma}_b{}^d - \frac{1}{3}\tilde{\gamma}_{ab}\tilde{\gamma}^{cd}\right)\left(\mathring{D}^c\mathring{D}_c - \Omega^2\partial_\phi^2\right)h_{cd} \\
&\qquad = -\frac{1}{2}\left[\left(\mathring{D}^c\mathring{D}_c - \Omega^2\partial_\phi^2\right)h_{ab} + \frac{1}{3}\mathring{D}^e h^{cd}\,\mathring{D}_e h_{cd} - \frac{1}{3}\Omega^2\partial_\phi h^{cd}\partial_\phi h_{cd}\right]
\end{aligned} \tag{188}
$$

8. *Helmholtz equation for h_{ab}*

Finally, the trace free operation results in the following Helmholtz equation,

$$
\left(\mathring{\Delta} - \Omega^2\partial_\phi^2\right)h_{ab} = 2\left(\bar{\mathcal{E}}_{ab} - \frac{1}{3}\tilde{\gamma}_{ab}\tilde{\gamma}^{cd}\bar{\mathcal{E}}_{cd}\right) - \frac{1}{3}\tilde{\gamma}_{ab}\mathring{D}^e h^{cd}\mathring{D}_e h_{cd} + \frac{1}{3}\tilde{\gamma}_{ab}\Omega^2\partial_\phi h^{cd}\partial_\phi h_{cd}. \tag{189}
$$

where $\bar{\mathcal{E}}_{ab}$ is defined by

$$
\bar{\mathcal{E}}_{ab} := \frac{1}{\alpha}\pounds_\omega K_{ab} - \frac{1}{2}\Omega^2\partial_\phi^2 h_{ab} + R_{ab}^{\rm NL} + {}^3\bar{R}_{ab}^{\psi} - \frac{1}{\alpha}\bar{D}_a\bar{D}_b\alpha + KK_{ab} - 2K_{ac}K_b{}^c - 8\pi\bar{S}_{ab}, \tag{190}
$$

or using the tracefree A_{ab}, $K_{ab} = A_{ab} + \frac{1}{3}\gamma_{ab}K$, and further drop the terms proportional to γ_{ab},

$$
\bar{\mathcal{E}}_{ab} := \frac{1}{\alpha}\pounds_\omega A_{ab} - \frac{1}{2}\Omega^2\partial_\phi^2 h_{ab} + R_{ab}^{\rm NL} + {}^3\bar{R}_{ab}^{\psi} - \frac{1}{\alpha}\bar{D}_a\bar{D}_b\alpha + \frac{1}{3}KA_{ab} - 2A_{ac}A_b{}^c - 8\pi\bar{S}_{ab}. \tag{191}
$$

Using the normalized $\tilde{A}_{ab}$

$$
\bar{\mathcal{E}}_{ab} := \frac{1}{\alpha}\pounds_\omega(\psi^4\tilde{A}_{ab}) - \frac{1}{2}\Omega^2\partial_\phi^2 h_{ab} + R_{ab}^{\rm NL} + {}^3\bar{R}_{ab}^{\psi} - \frac{1}{\alpha}\bar{D}_a\bar{D}_b\alpha + \frac{1}{3}\psi^4K\tilde{A}_{ab} - 2\psi^4\tilde{A}_{ac}\tilde{A}_b{}^c - 8\pi\bar{S}_{ab}, \tag{192}
$$

and with the maximal slice $K = 0$

$$
\bar{\mathcal{E}}_{ab} := \frac{1}{\alpha}\pounds_\omega(\psi^4\tilde{A}_{ab}) - \frac{1}{2}\Omega^2\partial_\phi^2 h_{ab} + R_{ab}^{\rm NL} + {}^3\bar{R}_{ab}^{\psi} - \frac{1}{\alpha}\bar{D}_a\bar{D}_b\alpha - 2\psi^4\tilde{A}_{ac}\tilde{A}_b{}^c - 8\pi\bar{S}_{ab}. \tag{193}
$$

In the above we used relations

$$
\pounds_n K_{ab} = \pounds_n\left(A_{ab} + \frac{1}{3}\gamma_{ab}K\right) = \pounds_n A_{ab} + \frac{1}{3}K\pounds_n\gamma_{ab} + \frac{1}{3}\gamma_{ab}\pounds_n K = \pounds_n A_{ab} - \frac{2}{3}KK_{ab} + \frac{1}{3}\gamma_{ab}\pounds_n K, \tag{194}
$$

which implies

$$
\frac{1}{\alpha}\pounds_\omega K_{ab} = \frac{1}{\alpha}\pounds_\omega A_{ab} + \frac{2}{3}K\left(A_{ab} + \frac{1}{3}\gamma_{ab}K\right) + \frac{1}{3\alpha}\gamma_{ab}\pounds_\omega K, \tag{195}
$$

and

$$
K_{ac}K_b{}^c = \left(A_{ac} + \frac{1}{3}\gamma_{ac}K\right)\left(A_b{}^c + \frac{1}{3}\gamma_b{}^cK\right) = A_{ac}A_b{}^c + \frac{2}{3}KA_{ab} + \frac{1}{9}\gamma_{ab}K^2. \tag{196}
$$

The fact that the trace free operation to the unbarred $\mathcal{E}_{ab}$ and barred $\bar{\mathcal{E}}_{ab}$ are identical, and hence a part proportional to $\tilde{\gamma}_{ab}$ in unbarred $\mathcal{E}_{ab}$ is dropped in barred $\bar{\mathcal{E}}_{ab}$.

A longer form is corresponds to Eq. (173) is written

$$
\begin{aligned}
\bar{\mathcal{E}}_{ab} \;:=\;& R^{\rm NL}_{ab} + {}^3\bar{R}^{\psi}_{ab} - \frac{1}{\alpha}\bar{D}_a\bar{D}_b\alpha + KK_{ab} - 2K_{ac}K_b{}^c - 8\pi\bar{S}_{ab} \\
&+\frac{1}{2}\left(\frac{\psi^4}{\alpha^2}-1\right)\Omega^2\partial_\phi^2 h_{ab} \;+\; \frac{\psi^4}{2\alpha^2}\,\Omega^2\left[\left(\partial_\phi\tilde{\gamma}_{ac}+\pounds_\phi\tilde{\gamma}_{ac}\right)\mathring{D}_b\phi^c \;+\; \left(\partial_\phi\tilde{\gamma}_{cb}+\pounds_\phi\tilde{\gamma}_{cb}\right)\mathring{D}_a\phi^c\right] \\
&+\frac{\psi^4}{2\alpha^2}\,\Omega\left(\pounds_\phi\pounds_\beta\tilde{\gamma}_{ab}+\pounds_\beta\pounds_\phi\tilde{\gamma}_{ab}\right) \;+\; \frac{\psi^4}{2\alpha^2}\,\pounds_\beta\pounds_\beta\tilde{\gamma}_{ab} \;+\; \frac{1}{\alpha}K_{ab}\,\pounds_\omega\ln\frac{\psi^8}{\alpha},
\end{aligned}
\tag{197}
$$

or with the tracefree $\tilde{A}_{ab}$ and dropping terms proportional to γ_{ab}

$$
\begin{aligned}
\bar{\mathcal{E}}_{ab} \;:=\;& R^{\rm NL}_{ab} + {}^3\bar{R}^{\psi}_{ab} - \frac{1}{\alpha}\bar{D}_a\bar{D}_b\alpha - \frac{1}{3}KA_{ab} - 2A_{ac}A_b{}^c - 8\pi\bar{S}_{ab} \\
&+\frac{1}{2}\left(\frac{\psi^4}{\alpha^2}-1\right)\Omega^2\partial_\phi^2 h_{ab} \;+\; \frac{\psi^4}{2\alpha^2}\,\Omega^2\left[\left(\partial_\phi\tilde{\gamma}_{ac}+\pounds_\phi\tilde{\gamma}_{ac}\right)\mathring{D}_b\phi^c \;+\; \left(\partial_\phi\tilde{\gamma}_{cb}+\pounds_\phi\tilde{\gamma}_{cb}\right)\mathring{D}_a\phi^c\right] \\
&+\frac{\psi^4}{2\alpha^2}\,\Omega\left(\pounds_\phi\pounds_\beta\tilde{\gamma}_{ab}+\pounds_\beta\pounds_\phi\tilde{\gamma}_{ab}\right) \;+\; \frac{\psi^4}{2\alpha^2}\,\pounds_\beta\pounds_\beta\tilde{\gamma}_{ab} \;+\; \frac{1}{\alpha}A_{ab}\,\pounds_\omega\ln\frac{\psi^8}{\alpha}.
\end{aligned}
\tag{198}
$$

With normalized $\tilde{A}_{ab}$ those are rewritten

$$
\begin{aligned}
\bar{\mathcal{E}}_{ab} \;:=\;& R^{\rm NL}_{ab} + {}^3\bar{R}^{\psi}_{ab} - \frac{1}{\alpha}\bar{D}_a\bar{D}_b\alpha - \frac{1}{3}\psi^4KA_{ab} - 2\psi^4\tilde{A}_{ac}\tilde{A}_b{}^c - 8\pi\bar{S}_{ab} \\
&+\frac{1}{2}\left(\frac{\psi^4}{\alpha^2}-1\right)\Omega^2\partial_\phi^2 h_{ab} \;+\; \frac{\psi^4}{2\alpha^2}\,\Omega^2\left[\left(\partial_\phi\tilde{\gamma}_{ac}+\pounds_\phi\tilde{\gamma}_{ac}\right)\mathring{D}_b\phi^c \;+\; \left(\partial_\phi\tilde{\gamma}_{cb}+\pounds_\phi\tilde{\gamma}_{cb}\right)\mathring{D}_a\phi^c\right] \\
&+\frac{\psi^4}{2\alpha^2}\,\Omega\left(\pounds_\phi\pounds_\beta\tilde{\gamma}_{ab}+\pounds_\beta\pounds_\phi\tilde{\gamma}_{ab}\right) \;+\; \frac{\psi^4}{2\alpha^2}\,\pounds_\beta\pounds_\beta\tilde{\gamma}_{ab} \;+\; \frac{\psi^4}{\alpha}\tilde{A}_{ab}\,\pounds_\omega\ln\frac{\psi^8}{\alpha},
\end{aligned}
\tag{199}
$$

and in maximal slice,

$$
\begin{aligned}
\bar{\mathcal{E}}_{ab} \;:=\;& R^{\rm NL}_{ab} + {}^3\bar{R}^{\psi}_{ab} - \frac{1}{\alpha}\bar{D}_a\bar{D}_b\alpha - 2\psi^4\tilde{A}_{ac}\tilde{A}_b{}^c - 8\pi\bar{S}_{ab} \\
&+\frac{1}{2}\left(\frac{\psi^4}{\alpha^2}-1\right)\Omega^2\partial_\phi^2 h_{ab} \;+\; \frac{\psi^4}{2\alpha^2}\,\Omega^2\left[\left(\partial_\phi\tilde{\gamma}_{ac}+\pounds_\phi\tilde{\gamma}_{ac}\right)\mathring{D}_b\phi^c \;+\; \left(\partial_\phi\tilde{\gamma}_{cb}+\pounds_\phi\tilde{\gamma}_{cb}\right)\mathring{D}_a\phi^c\right] \\
&+\frac{\psi^4}{2\alpha^2}\,\Omega\left(\pounds_\phi\pounds_\beta\tilde{\gamma}_{ab}+\pounds_\beta\pounds_\phi\tilde{\gamma}_{ab}\right) \;+\; \frac{\psi^4}{2\alpha^2}\,\pounds_\beta\pounds_\beta\tilde{\gamma}_{ab} \;+\; \frac{\psi^4}{\alpha}\tilde{A}_{ab}\,\pounds_\omega\ln\frac{\psi^8}{\alpha}.
\end{aligned}
\tag{200}
$$

E. comparison with the equation for the waveless formalism

In the waveless formalism, the timederivative of the conformal metric, $\partial_t\tilde\gamma_{ab}$ is truncated. As a reuslt the extrinxic curvature relates to the non-rotating shift β^a instead of the rotating shift ω^a,

$$K_{ab} = -\frac{1}{2}\pounds_n\gamma_{ab} \;=\; \frac{1}{2\alpha}\pounds_\beta\gamma_{ab} - \frac{1}{3\alpha}\gamma_{ab}\frac{1}{\sqrt{\gamma}}\pounds_t\sqrt{\gamma} \;=\; \frac{1}{2\alpha}\pounds_\beta\gamma_{ab} + \frac{1}{3\alpha}\gamma_{ab}D_c(\Omega\phi^c) \tag{201}$$

where the helical symmetry is used in the last equation. In $K=0$ gauge, the above is rewritten

$$K_{ab} = \frac{1}{2\alpha}\pounds_\beta\gamma_{ab}. \tag{202}$$

For the time derivative of K_{ab}, the helical symmetry is applied, and separating the laplacian, we have

$$\overset{\circ}{\Delta} h_{ab} = 2\left(\bar{\mathcal{E}}_{ab} - \frac{1}{3}\tilde\gamma_{ab}\tilde\gamma^{cd}\bar{\mathcal{E}}_{cd}\right) - \frac{1}{3}\tilde\gamma_{ab}\overset{\circ}{D}{}^e h^{cd}\overset{\circ}{D}_e h_{cd}, \tag{203}$$

where the source is written

$$\bar{\mathcal{E}}_{ab} := \frac{1}{\alpha}\pounds_\omega K_{ab} + R^{\rm NL}_{ab} + {}^3\bar R^{\psi}_{ab} - \frac{1}{\alpha}\bar D_a\bar D_b\alpha + KK_{ab} - 2K_{ac}K_b{}^c - 8\pi\bar S_{ab}, \tag{204}$$

or using tracefree A_{ab},

$$\bar{\mathcal{E}}_{ab} \;:=\; \frac{1}{\alpha}\pounds_\omega A_{ab} + R^{\rm NL}_{ab} + {}^3\bar R^{\psi}_{ab} - \frac{1}{\alpha}\bar D_a\bar D_b\alpha + \frac{1}{3}KA_{ab} - 2A_{ac}A_b{}^c - 8\pi\bar S_{ab}. \tag{205}$$

With the normalized $\tilde A_{ab}$,

$$\bar{\mathcal{E}}_{ab} \;:=\; \frac{1}{\alpha}\pounds_\omega(\psi^4\tilde A_{ab}) + R^{\rm NL}_{ab} + {}^3\bar R^{\psi}_{ab} - \frac{1}{\alpha}\bar D_a\bar D_b\alpha + \frac{1}{3}\psi^4K\tilde A_{ab} - 2\psi^4\tilde A_{ac}\tilde A_b{}^c - 8\pi\bar S_{ab}. \tag{206}$$

and in the maximal slicing

$$\bar{\mathcal{E}}_{ab} := \frac{1}{\alpha}\pounds_\omega(\psi^4\tilde A_{ab}) + R^{\rm NL}_{ab} + {}^3\bar R^{\psi}_{ab} - \frac{1}{\alpha}\bar D_a\bar D_b\alpha - 2\psi^4\tilde A_{ac}\tilde A_b{}^c - 8\pi\bar S_{ab}. \tag{207}$$

In Eq.(204), the first term appears instead of the last two lines in Eq.(197).

Because of the above difference in the extrinsic curvature, the momementum constraint is written differently. For the waveless spacetime, $(\tilde L\phi)_{ab}$ term is truncated in Eq.(107) and hence in Eq.(114), $\Omega\tilde D_b(\tilde L\phi)_a{}^b$ term is eliminated:

$$\tilde D_b\tilde D^b\tilde\beta_a + \frac{1}{3}\tilde D_a\tilde D_b\tilde\beta^b + {}^3\tilde R_{ab}\tilde\beta^b + 2\alpha\tilde A_a{}^b\frac{\alpha}{\psi^6}\tilde D_b\left(\frac{\psi^6}{\alpha}\right) = 16\pi\alpha j_a, \tag{208}$$

and similaly in (128).

F. Components in sphrical coordinate

One can introduce a chart with respect to the flat metric

$$dl^2 = f_{ab} dx^a dx^b, \tag{209}$$

Spherical coordiante (r, θ, ϕ), $r \in [0, \infty)$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$. In this chart, one may choose a set of cooridante basis $\{r^a, \theta^a, \phi^a\}$ or non-coordinate orthonormal basis $\{\hat{r}^a, \hat{\theta}^a, \hat{\phi}^a\}$.

1. Coordinate basis

Coordinate basis are defined by

$$r_a := \mathring{D}_a r, \quad \theta_a := \mathring{D}_a \theta, \quad \phi_a := \mathring{D}_a \phi. \tag{210}$$

and $\{r^a, \theta^a, \phi^a\}$, which are normalized by

$$r^a \mathring{D}_a r = 1, \quad \theta^a \mathring{D}_a \theta = 1, \quad \phi^a \mathring{D}_a \phi = 1. \tag{211}$$

and $r^a \theta_a = 0$, $r^a \phi_a = 0$, $\theta^a r_a = 0$, $\theta^a \phi_a = 0$, $\phi^a r_a = 0$, $\phi^a \theta_a = 0$. For these basis,

$$f_{ab} = \mathring{D}_a r \mathring{D}_b r + r^2 \mathring{D}_a \theta \mathring{D}_b \theta + r^2 \sin^2\theta \, \mathring{D}_a \phi \mathring{D}_b \phi, \tag{212}$$

and a metric on a 2-sphere with radius r (a projection tensor) is written

$$\begin{aligned} e_{ab} &= \frac{1}{r^2} \left(f_{ab} - \mathring{D}_a r \mathring{D}_b r \right) = \frac{1}{r^2} \left(f_{ab} - r_a r_b \right) & (213) \\ &= \mathring{D}_a \theta \mathring{D}_b \theta + \sin^2\theta \, \mathring{D}_a \phi \mathring{D}_b \phi, & (214) \\ e^{ab} &= \theta^a \theta^b + \frac{1}{\sin^2\theta} \phi^a \phi^b, & (215) \end{aligned}$$

which has properties $r^a e_{ab} = 0$, $r_a e^{ab} = 0$, and $e^{ac} e_{cb} = \theta^a \mathring{D}_b \theta + \phi^a \mathring{D}_b \phi = e^a{}_b$.

For the coordinate basis, one can not lower index of basis by the flat metric f_{ab}.

2. Non-coordinate orthonormal basis

Normalized basis $\{\hat{r}^a, \hat{\theta}^a, \hat{\phi}^a\}$, $\hat{r}_a = f_{ab} \hat{r}^b$, $\hat{\theta}_a = f_{ab} \hat{\theta}^b$, $\hat{\phi}_a = f_{ab} \hat{\phi}^b$, ($f_{ab} = \mathrm{diag}(1, 1, 1)$, hence $\hat{r}_a = \hat{r}^a$, $\hat{\theta}_a = \hat{\theta}^a$, $\hat{\phi}_a = \hat{\phi}^a$.)

$$\hat{r}^a = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \; \hat{\theta}^a = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \; \hat{\phi}^a = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}. \tag{216}$$

For these basis,

$$f_{ab} = \hat{r}_a \hat{r}_b + \hat{\theta}_a \hat{\theta}_b + \hat{\phi}_a \hat{\phi}_b. \tag{217}$$

The metric on an unit 2-sphere $\hat{e}_{ab}$ is written

$$\begin{aligned} \hat{e}_{ab} &= f_{ab} - \mathring{D}_a r \mathring{D}_b r = f_{ab} - \hat{r}_a \hat{r}_b & (218) \\ &= \hat{\theta}_a \hat{\theta}_b + \hat{\phi}_a \hat{\phi}_b, & (219) \\ \hat{e}^{ab} &= \hat{\theta}^a \hat{\theta}^b + \hat{\phi}^a \hat{\phi}^b, & (220) \end{aligned}$$

which has properties $\hat{r}^a \hat{e}_{ab} = 0$, $\hat{r}_a \hat{e}^{ab} = 0$, and $\hat{e}^{ac} \hat{e}_{cb} = \hat{\theta}^a \hat{\theta}_b + \hat{\phi}^a \hat{\phi}_b = \hat{e}^a{}_b$. For the non-coordinate orthnormal basis, index is lowered by the flat metric f_{ab}.

3. Relations between two basis

$$r_a = \mathring{D}_a r = \hat{r}_a, \qquad r^a = \hat{r}^a, \tag{221}$$
$$\theta_a = \mathring{D}_a \theta = \frac{1}{r}\hat{\theta}_a, \qquad \theta^a = r\hat{\theta}^a, \tag{222}$$
$$\phi_a = \mathring{D}_a \phi = \frac{1}{r\sin\theta}\hat{\phi}_a, \qquad \phi^a = r\sin\theta\hat{\phi}^a, \tag{223}$$
$$e_{ab} = \frac{\hat{e}_{ab}}{r^2}, \qquad e^{ab} = r^2\hat{e}^{ab}, \tag{224}$$

4. Useful formulas for the non-coordinate orthonormal basis

Derivatives :

$$\frac{\partial}{\partial r}\hat{r}_a = 0, \qquad \frac{\partial}{\partial\theta}\hat{r}_a = \hat{\theta}_a, \qquad \frac{\partial}{\partial\phi}\hat{r}_a = \sin\theta\,\hat{\phi}_a, \tag{225}$$
$$\frac{\partial}{\partial r}\hat{\theta}_a = 0, \qquad \frac{\partial}{\partial\theta}\hat{\theta}_a = -\hat{r}_a, \qquad \frac{\partial}{\partial\phi}\hat{\theta}_a = \cos\theta\,\hat{\phi}_a, \tag{226}$$
$$\frac{\partial}{\partial r}\hat{\phi}_a = 0, \qquad \frac{\partial}{\partial\theta}\hat{\phi}_a = 0, \qquad \frac{\partial}{\partial\phi}\hat{\phi}_a = -\sin\theta\,\hat{r}_a - \cos\theta\,\hat{\theta}_a. \tag{227}$$

Gradient:

$$\mathring{D}_a := \hat{r}_a\frac{\partial}{\partial r} + \hat{\theta}_a\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}_a\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}. \tag{228}$$

$$\begin{aligned}
\mathring{D}_a\hat{r}_b &= \frac{1}{r}\left(\hat{\theta}_a\hat{\theta}_b + \hat{\phi}_a\hat{\phi}_b\right) = \frac{\hat{e}_{ab}}{r}, && (229)\\
\mathring{D}_a\hat{\theta}_b &= -\frac{1}{r}\hat{\theta}_a\hat{r}_b + \frac{1}{r\tan\theta}\hat{\phi}_a\hat{\phi}_b, && (230)\\
\mathring{D}_a\hat{\phi}_b &= -\frac{1}{r}\hat{\phi}_a\hat{r}_b - \frac{1}{r\tan\theta}\hat{\phi}_a\hat{\theta}_b, && (231)\\
\mathring{D}_a f_{bc} &= 0. && (232)
\end{aligned}$$

($\hat{r}^a\hat{r}_a = 1$ implies $\hat{r}^a\mathring{D}_b\hat{r}_a = \hat{r}_a\mathring{D}_b\hat{r}^a = 0$. Since $\hat{r}^a\mathring{D}_a\hat{r}_b = \hat{r}^a\mathring{D}_a\mathring{D}_b r = \hat{r}^a\mathring{D}_b\hat{r}_a = 0$, $\mathring{D}_a\hat{r}_b \propto \hat{e}_{ab}$. With $\mathring{D}_a\hat{r}^a = 2/r$, we have $\mathring{D}_a\hat{r}_b = \hat{e}_{ab}/r$.)

For the metric on 2-sphere,

$$\begin{aligned}
\mathring{D}_a\hat{e}_{bc} &= -\mathring{D}_a(\hat{r}_b\hat{r}_c) = -\frac{1}{r}(\hat{e}_{ab}\hat{r}_c + \hat{e}_{ac}\hat{r}_b), && (233)\\
\mathring{D}_b\hat{e}_a{}^b &= -\frac{2}{r}\hat{r}_a. && (234)
\end{aligned}$$

For the Lie derivative with respect to $\hat{r}^a$, $\hat{\theta}^a$ and $\hat{\phi}^a$:

$$\pounds_{\hat{r}}\hat{r}^a = 0, \qquad \pounds_{\hat{r}}\hat{\theta}^a = -\frac{1}{r}\hat{\theta}^a, \qquad \pounds_{\hat{r}}\hat{\phi}^a = -\frac{1}{r}\hat{\phi}^a, \tag{235}$$
$$\pounds_{\hat{\theta}}\hat{r}^a = \frac{1}{r}\hat{\theta}^a, \qquad \pounds_{\hat{\theta}}\hat{\theta}^a = 0, \qquad \pounds_{\hat{\theta}}\hat{\phi}^a = -\frac{1}{r\tan\theta}\hat{\phi}^a, \tag{236}$$
$$\pounds_{\hat{\phi}}\hat{r}^a = \frac{1}{r}\hat{\phi}^a, \qquad \pounds_{\hat{\phi}}\hat{\theta}^a = \frac{1}{r\tan\theta}\hat{\phi}^a, \qquad \pounds_{\hat{\phi}}\hat{\phi}^a = 0. \tag{237}$$

5. Useful formulas for the coordinate basis

Derivatives :

$$\frac{\partial}{\partial r} r^a = 0, \qquad \frac{\partial}{\partial \theta} r^a = \frac{1}{r}\theta^a, \qquad \frac{\partial}{\partial \phi} r^a = \frac{1}{r}\phi^a, \tag{238}$$

$$\frac{\partial}{\partial r}\theta^a = \frac{1}{r}\theta^a, \qquad \frac{\partial}{\partial \theta}\theta^a = -r\, r^a, \qquad \frac{\partial}{\partial \phi}\theta^a = \frac{1}{\tan\theta}\,\phi^a, \tag{239}$$

$$\frac{\partial}{\partial r}\phi^a = \frac{1}{r}\phi^a, \qquad \frac{\partial}{\partial \theta}\phi^a = \frac{1}{\tan\theta}\,\phi^a, \qquad \frac{\partial}{\partial \phi}\phi^a = -r\sin^2\theta\, r^a - \sin\theta\cos\theta\,\theta^a. \tag{240}$$

Gradient:

$$\mathring{D}_a = \mathring{D}_a r \frac{\partial}{\partial r} + \mathring{D}_a \theta \frac{\partial}{\partial \theta} + \mathring{D}_a \phi \frac{\partial}{\partial \phi} \tag{241}$$

$$\begin{aligned}
\mathring{D}_a r^b &= \frac{1}{r}\left(\mathring{D}_a\theta\,\theta^b + \mathring{D}_a\phi\,\phi^b\right) && (242)\\
&= \frac{1}{r}\left(\hat\theta_a\,\hat\theta^b + \hat\phi_a\,\hat\phi^b\right) = \frac{1}{r} e_a{}^b, && (243)\\
\mathring{D}_a\theta^b &= \frac{1}{r}\mathring{D}_a r\,\theta^b - r\mathring{D}_a\theta\, r^b + \frac{1}{\tan\theta}\mathring{D}_a\phi\,\phi^b && (244)\\
&= \left(\hat r_a\,\hat\theta^b - \hat\theta_a\, r^b\right) + \frac{1}{\tan\theta}\hat\phi_a\,\hat\phi^b, && (245)\\
\mathring{D}_a\phi^b &= \frac{1}{r}\mathring{D}_a r\,\phi^b - r\sin^2\theta\mathring{D}_a\phi\, r^b + \frac{1}{\tan\theta}\mathring{D}_a\theta\,\phi^b - \sin\theta\cos\theta\mathring{D}_a\phi\,\theta^b && (246)\\
&= \sin\theta\left(\hat r_a\,\hat\phi^b - \hat\phi_a\,\hat r^b\right) + \cos\theta\left(\hat\theta_a\,\hat\phi^b - \hat\phi_a\,\hat\theta^b\right). && (247)
\end{aligned}$$

Lie derivatives:

$$\pounds_X Y^a = 0 \quad \text{for all coordinate basis} \quad X^a, Y^a = r^a, \theta^a, \phi^a. \tag{248}$$

For a function f, and fields X^a and Y^a

$$\pounds_{fX} Y^a = f\pounds_X Y^a - X^a \pounds_Y f, \tag{249}$$

$$\pounds_{fX} Y_a = f\pounds_X Y_a + X^b Y_b \mathring{D}_a f, \tag{250}$$

$$\pounds_X (fY^a) = f\pounds_X Y^a + Y^a \pounds_X f \tag{251}$$

$$\pounds_X (fY_a) = f\pounds_X Y_a + Y_a \pounds_X f. \tag{252}$$

$$\pounds_r \hat r^a = 0, \qquad \pounds_r \hat\theta^a = -\frac{1}{r}\hat\theta^a, \qquad \pounds_r \hat\phi^a = -\frac{1}{r}\hat\phi^a, \tag{253}$$

$$\pounds_\theta \hat r^a = 0, \qquad \pounds_\theta \hat\theta^a = 0, \qquad \pounds_\theta \hat\phi^a = -\frac{1}{\tan\theta}\hat\phi^a, \tag{254}$$

$$\pounds_\phi \hat r^a = 0, \qquad \pounds_\phi \hat\theta^a = 0, \qquad \pounds_\phi \hat\phi^a = 0. \tag{255}$$

$$\pounds_r \hat r_a = 0, \qquad \pounds_r \hat\theta_a = \frac{1}{r}\hat\theta_a, \qquad \pounds_r \hat\phi_a = \frac{1}{r}\hat\phi_a, \tag{256}$$

$$\pounds_\theta \hat r_a = 0, \qquad \pounds_\theta \hat\theta_a = 0, \qquad \pounds_\theta \hat\phi_a = \frac{1}{\tan\theta}\hat\phi_a, \tag{257}$$

$$\pounds_\phi \hat r_a = 0, \qquad \pounds_\phi \hat\theta_a = 0, \qquad \pounds_\phi \hat\phi_a = 0. \tag{258}$$

$$\pounds_{\hat{r}} r^a = 0, \qquad \pounds_{\hat{r}} \theta^a = 0, \qquad \pounds_{\hat{r}} \phi^a = 0, \tag{259}$$

$$\pounds_{\hat{\theta}} r^a = \frac{1}{r}\hat{\theta}^a, \qquad \pounds_{\hat{\theta}} \theta^a = 0, \qquad \pounds_{\hat{\theta}} \phi^a = 0, \tag{260}$$

$$\pounds_{\hat{\phi}} r^a = \frac{1}{r}\hat{\phi}^a, \qquad \pounds_{\hat{\phi}} \theta^a = \frac{1}{\tan\theta}\hat{\phi}^a, \qquad \pounds_{\hat{\phi}} \phi^a = 0. \tag{261}$$

$$\pounds_r f_{ab} = \frac{2}{r}(\hat{\theta}_a\hat{\theta}_b + \hat{\phi}_a\hat{\phi}_b) = \frac{2}{r}\hat{e}_{ab}, \tag{262}$$

$$\pounds_\theta f_{ab} = \frac{2}{\tan\theta}\phi_a\phi_b, \tag{263}$$

$$\pounds_\phi f_{ab} = 0. \tag{264}$$

IV. EQUATIONS FOR THE FLUID

The helical vector is defined as

$$k^\alpha = t^\alpha + \Omega\phi^\alpha, \tag{265}$$

where ϕ^α is a spatial vector field which generates a family of closed circular curves on Σ_t, and Ω denotes the orbital angular velocity, Here, a spatial velocity vector v^α is introduced as

$$u^\alpha = u^t(k^\alpha + v^\alpha). \tag{266}$$

For an isentropic fluid, conservation of the rest mass and the entropy,

$$£_u(\rho\sqrt{-g}) = 0, \quad \text{and} \quad £_u s = 0, \tag{267}$$

or

$$\nabla_\alpha(\rho u^\alpha) = 0, \quad \text{and} \quad u^\alpha\nabla_\alpha s = 0. \tag{268}$$

For the isentropic flow, the local first law of the thermodynamics becomes

$$\frac{1}{\rho}\nabla_\alpha p = \nabla_\alpha h. \tag{269}$$

Equation (269) also implies that one parameter equation of state may be chosen. Thus, we have four independent variables for the fluid, one for the thermodynamic variable and three for the fluid velocity, which are governed by four equations (271) and (270).

The independent variation of displacement vector ξ^a results in the relativistic Euler equation,

$$\partial_t(j_a\sqrt{\gamma}) - £_\beta(j_a\sqrt{\gamma}) + D_b(S_a{}^b\alpha\sqrt{\gamma}) + \rho_H D_a\alpha\sqrt{\gamma} = 0\,. \tag{270}$$

The baryon mass conservation law Eq. (267) and the Euler equation Eq. (270) with (268) becomes

$$£_{k+v}(\rho u^t\sqrt{-g}) = 0, \tag{271}$$

$$\gamma_a{}^\alpha £_{k+v}(hu_\alpha) \;+\; D_a\left(\frac{h}{u^t}\right) = 0. \tag{272}$$

Then, we impose the conditions that the Lie derivatives along k^α vanish

$$£_k(\rho u^t\sqrt{-g}) = 0, \tag{273}$$

$$\gamma_a{}^\alpha £_k(hu_\alpha) = 0. \tag{274}$$

As a result of the conditions (273) and (274), a relation $£_k(j_a\sqrt{\gamma}) = 0$ is also obtained.

For corotational flow $u^\alpha = u^t k^\alpha$ (that is $v^\alpha = 0$) or irrotational flow $hu_\alpha = \nabla_\alpha\Phi$, where Φ is a velocity potential, one can obtain the first integral of the Euler equation which is suitable for computing quasiequilibrium configurations. For corotational flow, the velocity field becomes trivial and the thermodynamic variable is evaluated from the relativistic Bernoulli's equation as

$$\frac{h}{u^t} = \text{constant}. \tag{275}$$

For the irrotational flow, one thermodynamic variable and a velocity potential are governed by two equations,

$$D_a\left[\frac{\alpha\rho}{h}\left(D^a\Phi - hu^t\omega^a\right)\right] = 0, \tag{276}$$

$$£_v\Phi \;+\; \frac{h}{u^t} = \text{constant}, \tag{277}$$

which are derived from Eqs. (271) and (270), respectively. Here, the Lie derivative with respect to the spatial vector v^a, $£_v$, is defined on Σ_t with a relation

$$v^a = \gamma^a{}_\alpha\left(\frac{u^\alpha}{u^t} - \omega^\alpha\right) = \frac{1}{hu^t}D^a\Phi - \omega^a, \tag{278}$$

where ω^α is a rotational shift vector defined as $\omega^\alpha = \beta^\alpha + \Omega\phi^\alpha$.

V. ASYMPTOTICS

A. Radiative boundary condition for h_{ab}

Along approximate light cone coordinates of the incoming and outgoing null geodesics, $u_\pm = \alpha t \pm \psi^2 r$ (or $u_\pm = t \pm r$ may be accurate enough), the metric components h_{ab} approximately satisfy

$$h_{ab}(u_\pm) = \frac{C_{ab}}{r}, \tag{279}$$

where C_{ab} is a constant matrix. Therefore, a relation

$$r\, h_{ab}(t,r) = (r \pm \delta r)\, h_{ab}(t \pm \delta t, r \pm \delta r), \tag{280}$$

results in

$$\partial_t h_{ab} + \frac{\delta r}{\delta t}\left(\partial_r h_{ab} + \frac{1}{r} h_{ab}\right) = 0. \tag{281}$$

Along the light cones, $u_\pm =$ constant, and hence for the changes in $t \to t \pm \delta t$ and $r \to r \pm \delta r$, we have

$$\alpha t \,\pm\, \psi^2 r = \alpha(t \pm \delta t) \,\pm\, \psi^2 (r \pm \delta r), \tag{282}$$

that is,

$$\frac{\delta r}{\delta t} = \mp \frac{\alpha}{\psi^2}, \tag{283}$$

B. Field equations in generalized Dirac gauge

Finally, the asymptotic conditions of the hypersurface Σ_t which is used as boundary conditions for the field equations are given as,

$$\alpha = 1 - \frac{\overset{1}{\alpha}}{r} + O(r^{-2}), \qquad \beta^a = O(r^{-2}), \quad \text{and} \quad \gamma_{ab} = f_{ab} + O(r^{-1}). \tag{284}$$

The above choice turned out to be consistent with the field equations, which is discussed in the later section.

A manifest waveless property is realized by introducing appropriate gauge conditions which cast the field equations into the elliptic type PDE. We follow a prescription often used in a formulation for computing numerical solutions of the astrophysical objects such as binary neutron stars.

The conformally related metric $\tilde\gamma_{ab} = \psi^{-4}\gamma_{ab}$, $\tilde\gamma_{ac}\tilde\gamma^{cb} = \delta_a{}^b$, and the conformal factor ψ, is introduced together with a gauge condition of either

$$\tilde\gamma = f \quad \text{or} \quad f^{ab}(\tilde\gamma_{ab} - f_{ab}) = 0. \tag{285}$$

This decomposition enable us to write the Hamiltonian constraint $\mathcal{H} = 0$ to be the elliptic equation for ψ since

$${}^3R = {}^3\tilde R\, \psi^{-4} - 8\psi^{-5} \tilde D^a \tilde D_a \psi \tag{286}$$

where $\tilde D_a$ and ${}^3\tilde R$ are the covariant derivative and the scalar curvature associated with $\tilde\gamma_{ab}$. A gauge condition $\pi = 0$ which has been imposed as in Eq. (??) guarantees the momentum constraint $\mathcal{C}_a = 0$ to be the elliptic equations for the shift vector β^a,

$$D_b \pi_a{}^b = \tilde D_b \tilde\pi_a{}^b = -\frac{\sqrt{\gamma}}{2\alpha}(\tilde D_b \tilde D^b \tilde\beta_a + \frac{1}{3}\tilde D_a \tilde D_b \tilde\beta^b + {}^3\tilde R_{ab}\tilde\beta^b) + \tilde\pi_a{}^b \tilde D_b \ln\frac{\sqrt{\gamma}}{2\alpha}\ , \tag{287}$$

where indices of tilded quantities $\tilde\pi^{ab}$ and $\tilde\beta^a$ are lowered and raised by $\tilde\gamma_{ab}$ and $\tilde\gamma^{ab}$ respectively, with a definition $\tilde\pi_a{}^b = \pi_a{}^b$ and $\tilde\beta^a = \beta^a$.

Since all the field equations are written manifestly in the elliptic form in this gauge, one may expect that the source terms of each equations decays fast enough and hence elliptic parts dominate at large r. The field variables becomes harmonic in the asymptotic region, namely,

$$\begin{aligned} \mathring{D}^a\mathring{D}_a\psi\,|_{r\to\infty} &= 0, \qquad \mathring{D}^a\mathring{D}_a\alpha\,|_{r\to\infty} = 0, \\ \left(\mathring{D}^b\mathring{D}_b\beta_a + \frac{1}{3}\mathring{D}_a\mathring{D}_b\beta^b\right)_{r\to\infty} &= 0, \qquad \mathring{D}^c\mathring{D}_c h_{ab}\,|_{r\to\infty} = 0, \end{aligned} \tag{288}$$

In this case, the field equations are solved with the same condition as Eq(284)

$$\begin{aligned} \alpha = 1 - \frac{\overset{1}{\alpha}}{r} + O(r^{-2}), &\qquad \psi = 1 + O(r^{-1}), \\ \beta^a = O(r^{-2}), &\qquad h_{ab} = O(r^{-1}). \end{aligned} \tag{289}$$

Since field variables are harmonic in the asymptotic region, the following conditions are also satisfied,

$$\begin{aligned} \mathring{D}_a\alpha = \frac{\overset{1}{\alpha}}{r^2} + O(r^{-3}), &\qquad \mathring{D}_a\psi = O(r^{-2}), \\ \mathring{D}_b\beta^a = O(r^{-3}), \qquad \mathring{D}_c h_{ab} \;=\; O(r^{-2}), &\qquad \mathring{D}_c\pi_{ab} = O(r^{-4}), \end{aligned} \tag{290}$$

which may be seen by expanding them using (tensor) spherical harmonics. The above conditions and Eq. (??) implies asymptotic behavior of π_{ab} as

$$\pi_{ab} = O(r^{-3})\,. \tag{291}$$

Also from Eqs. (289) and (290), and the Hamiltonian constraint (A13), the scalar curvatures ${}^3R$ and ${}^3\tilde{R}$ with respect to γ_{ab} and $\tilde{\gamma}_{ab}$, respectively, become

$${}^3R = O(r^{-4}) \;\text{ and }\; {}^3\tilde{R} = O(r^{-4}). \tag{292}$$

This may not contradict with Eq. (95) since ψ is a harmonic function asymptotically.

VI. ESTIMATION FOR THE VIOLATION OF HELICAL SYMMETRY

We estimate how much the solutions of the waveless or IWM formulations violate the helical symmetry by computing a norm of $\pounds_k K_{ab}$ which vanishes for the exactly helically symmetric solutions. Practically, $\pounds_k K_{ab}$ is calculated by substitutiing either the waveless solution $\{\alpha, \beta^a, \psi, \tilde{\gamma}_{ab}\}^{\rm WL}$ or the IWM solution $\{\alpha, \beta^a, \psi\}^{\rm IWM}$ to the Einstein equation on which the helical symmetry is imposed exactly.

For the helically symmetric spacetime with the metric $\{\alpha, \psi, \beta^a, \tilde{\gamma}_{ab}\}$, the extrinsic curvature is written

$$K_{ab} = -\frac{1}{2}\pounds_n \gamma_{ab} = -\frac{1}{2\alpha}\pounds_k \gamma_{ab} + \frac{1}{2\alpha}\pounds_\omega \gamma_{ab} = \frac{1}{2\alpha}\pounds_\omega \gamma_{ab}. \tag{293}$$

Writing Eq.(145), (144), and (175),

$$(G_{\alpha\beta} - 8\pi T_{\alpha\beta})(\gamma_a{}^\alpha \gamma_b{}^\beta - \frac{1}{3}\gamma_{ab}\gamma^{\alpha\beta}) = \mathcal{E}_{ab} - \frac{1}{3}\tilde{\gamma}_{ab}\tilde{\gamma}^{cd}\mathcal{E}_{cd} = 0, \tag{294}$$

$$\begin{aligned} \mathcal{E}_{ab} &:= -\pounds_n K_{ab} + {}^3R_{ab} + KK_{ab} - 2K_{ac}K_b{}^c - \frac{1}{\alpha}D_a D_b \alpha - 8\pi S_{ab} \\ &= -\frac{1}{\alpha}\pounds_k K_{ab} + \frac{1}{\alpha}\pounds_\omega K_{ab} + {}^3\tilde{R}_{ab} + {}^3\tilde{R}^\psi_{ab} - \frac{1}{\alpha}D_a D_b\alpha + KK_{ab} - 2K_{ac}K_b{}^c - 8\pi S_{ab}, \end{aligned} \tag{295}$$

the trace free part of the spatial projection of $\pounds_k K_{ab}$ is written in the maximal slice,

$$\begin{aligned} &(\gamma_a{}^c\gamma_b{}^d - \frac{1}{3}\gamma_{ab}\gamma^{cd})\pounds_k K_{cd} \\ &= (\tilde{\gamma}_a{}^c\tilde{\gamma}_b{}^d - \frac{1}{3}\tilde{\gamma}_{ab}\tilde{\gamma}^{cd})\left[\pounds_\omega(\psi^4\tilde{A}_{cd}) + \alpha\; {}^3\tilde{R}_{cd} + \alpha\left({}^3\tilde{R}^\psi_{cd} - \frac{1}{\alpha}D_c D_d \alpha\right) - 2\alpha\psi^4 \tilde{A}_{ce}\tilde{A}_d{}^e - 8\pi\alpha S_{cd}\right]. \end{aligned} \tag{296}$$

Note that the tracefree part of the extrinsic curvature A_{ab} is not that of the waveless (or IWM) K_{ab}, but the one defined by Eq. (293). Note also a relation,

$$\left(\tilde{\gamma}_a{}^c\tilde{\gamma}_b{}^d - \frac{1}{3}\tilde{\gamma}_{ab}\tilde{\gamma}^{cd}\right)\left({}^3\tilde{R}^\psi_{cd} - \frac{1}{\alpha}D_c D_d\alpha\right) \tag{297}$$

$$= \left(\tilde{\gamma}_a{}^c\tilde{\gamma}_b{}^d - \frac{1}{3}\tilde{\gamma}_{ab}\tilde{\gamma}^{cd}\right)\left\{-\frac{1}{\alpha\psi^2}\mathring{D}_c\mathring{D}_d(\alpha\psi^2) + \frac{1}{\alpha\psi^2}C^e_{cd}\mathring{D}_e(\alpha\psi^2) + \frac{4}{\alpha\psi^2}\left[\mathring{D}_c(\alpha\psi)\mathring{D}_d\psi + \mathring{D}_d(\alpha\psi)\mathring{D}_c\psi\right]\right\} \tag{298}$$

For the spatial trace part, writing (130) and (131),

$$(G_{\alpha\beta} - 8\pi T_{\alpha\beta})\gamma^{\alpha\beta} = 0 \tag{299}$$

$$(G_{\alpha\beta} - 8\pi T_{\alpha\beta})\gamma^{\alpha\beta} = 2\pounds_n K - \frac{1}{2}({}^3R + K^2 + 3K_{ab}K^{ab}) + \frac{2}{\alpha}D^a D_a\alpha - 8\pi S \tag{300}$$

$$\begin{aligned} &= \frac{2}{\alpha}\gamma^{ab}\pounds_k K_{ab} - \frac{2}{\alpha}\pounds_\omega K + 4K^{ab}\left(K_{ab} - \frac{1}{2\alpha}\pounds_\omega\gamma_{ab}\right) \\ &\quad + \left(-\frac{1}{2}{}^3R + \frac{2}{\alpha}D^a D_a\alpha\right) - \frac{1}{2}(K^2 + 3K_{ab}K^{ab}) - 8\pi S = 0, \end{aligned} \tag{301}$$

where we used

$$\pounds_n K = \gamma^{ab}\pounds_n K_{ab} + 2K_{ab}K^{ab} = \frac{1}{\alpha}\gamma^{ab}\pounds_k K_{ab} - \frac{1}{\alpha}\pounds_\omega K + 2K^{ab}\left(K_{ab} - \frac{1}{2\alpha}\pounds_\omega\gamma_{ab}\right). \tag{302}$$

Then using the helical K_{ab} Eq. (293) again, $\gamma^{ab}\pounds_k K_{ab}$ is written in $K=0$ gauge,

$$\gamma^{ab}\pounds_k K_{ab} = -\frac{\alpha}{2}\left(-\frac{1}{2}{}^3R + \frac{2}{\alpha}D^a D_a\alpha\right) + \frac{3}{4}\,\alpha\,\tilde{A}_{ab}\tilde{A}^{ab} + 4\pi\alpha S, \tag{303}$$

where

$$-\frac{1}{2}{}^3R + \frac{2}{\alpha}D^aD_a\alpha = -\frac{1}{2}\gamma^{ab}({}^3\tilde{R}_{ab} + {}^3\tilde{R}^{\psi}_{ab}) + \frac{2}{\alpha}D^aD_a\alpha \tag{304}$$

$$= -\frac{1}{2\psi^4}{}^3\tilde{R} + \frac{2}{\alpha\psi^6}\left[\tilde{D}^a\tilde{D}_a(\alpha\psi^2) - 2\tilde{D}_a(\alpha\psi)\tilde{D}^a\psi\right]. \tag{305}$$

Then, we introduce integrals of $£_k K_{ab}$ over the near zone to estimate the violation of the helical symmetry as follows;

$$||£_k K_{ab}||_1 := \int_{r\le r_0} \sqrt{\gamma^{ac}\gamma^{bd}£_k K_{ab}£_k K_{cd}}\sqrt{\gamma}d^3x, \tag{306}$$

$$||£_k K_{ab}||_2 := \sqrt{\int_{r\le r_0} \gamma^{ac}\gamma^{bd}£_k K_{ab}£_k K_{cd}\sqrt{\gamma}d^3x}, \tag{307}$$

where,

$$£_k K_{ab} = (\gamma_a{}^c\gamma_b{}^d - \frac{1}{3}\gamma_{ab}\gamma^{cd})£_k K_{cd} + \frac{1}{3}\gamma_{ab}\gamma^{cd}£_k K_{cd}. \tag{308}$$

For the above norms, the integarls are unbounded if one extend the integral to infinity, $r_0 \to \infty$. For the norm,

$$||£_k K_{ab}||_4 := \left[\int_{r\le r_0} \left(\gamma^{ac}\gamma^{bd}£_k K_{ab}£_k K_{cd}\right)^2 \sqrt{\gamma}d^3x\right]^{1/4}, \tag{309}$$

the integral is bounded for $r_0 \to \infty$. The waveless solution $\{\alpha, \beta^a, \psi, \tilde{\gamma}_{ab}\}^{\rm WL}$ or the IWM solution $\{\alpha, \beta^a, \psi\}^{\rm IWM}$ are substituted to the above $\{\alpha, \beta^a, \psi, \tilde{\gamma}_{ab}\}$, which is derived from the helically symmetric Einstein equations. The domain for integaration is taken from the center of orbit to the certain radius r_0 for which we take $r_0/(\pi/\Omega) = 0.1, 0.2, 0.5, 1.0$. Also compute the value over the fluid support and whole region.

N.B. 0.: These norms are normalized by the total gravitational mass of isolated binary neutron stars M_∞ as,

$$||£_k K_{ab}||_1/M_\infty, \quad ||£_k K_{ab}||_2\, M_\infty^{1/2}, \quad \text{and} \quad ||£_k K_{ab}||_4\, M_\infty^{5/4}. \tag{310}$$

N.B. 1.: Definitions for the extrinsic curvature of the helically symmetric spacetime and the waveless formulation (+ helical symmetry) differs as follows:

$$K_{ab} = \frac{1}{2\alpha}£_\omega\gamma_{ab}, \qquad \text{helical}, \tag{311}$$

$$K_{ab} = \frac{1}{2\alpha}£_\beta\gamma_{ab} + \frac{1}{6\alpha}\gamma_{ab}\gamma^{cd}£_{\Omega\phi}\gamma_{cd}, \qquad \text{waveless}. \tag{312}$$

However, the trace K are the same for both the helically symmetric spacetime and the waveless formulation,

$$K = \frac{1}{2\alpha}\gamma^{ab}£_\omega\gamma_{ab} = 0. \tag{313}$$

Therefore the maximal slice condition $K = 0$ takes the same form for both cases.

N.B. 2.: One can analytically subtract the waveless equation from the Eqs.(296) and (303) to calculate the violation of helical symmetry,

$$(\gamma_a{}^c\gamma_b{}^d - \frac{1}{3}\gamma_{ab}\gamma^{cd})£_k K_{cd} = (\tilde{\gamma}_a{}^c\tilde{\gamma}_b{}^d - \frac{1}{3}\tilde{\gamma}_{ab}\tilde{\gamma}^{cd})\left[£_\omega(\psi^4\bar{A}_{cd}) - 2\alpha\psi^4\bar{A}_{ce}\bar{A}_d{}^e,\right] \tag{314}$$

$$\gamma^{ab}£_k K_{ab} = \frac{3}{4}\,\alpha\,\bar{A}_{ab}\bar{A}^{ab}, \tag{315}$$

where

$$\bar{A}_{ab} := (\gamma_a{}^c\gamma_b{}^d - \frac{1}{3}\gamma_{ab}\gamma^{cd})\frac{1}{2\alpha}\left(£_\omega\tilde{\gamma}_{cd} - £_\beta\tilde{\gamma}_{cd}\right) = \frac{1}{2\alpha}\Omega(\tilde{L}\phi)_{ab}. \tag{316}$$

VII. ESTIMATION FOR THE VIOLATION OF HELICAL SYMMETRY (SHORT VERSION)

We estimate how much the solutions of the waveless or IWM formulations violate the helical symmetry by computing a norm of $\pounds_k K_{ab}$ which vanishes for the exactly helically symmetric solutions. Practically, $\pounds_k K_{ab}$ is calculated by substituting either the waveless solution $\{\alpha, \beta^a, \psi, \tilde{\gamma}_{ab}\}^{\rm WL}$ or the IWM solution $\{\alpha, \beta^a, \psi\}^{\rm IWM}$ to the Einstein equation on which the helical symmetry is imposed exactly.

For the helically symmetric spacetime with the metric $\{\alpha, \psi, \beta^a, \tilde{\gamma}_{ab}\}$, the extrinsic curvature is written

$$K_{ab} = -\frac{1}{2}\pounds_n \gamma_{ab} = -\frac{1}{2\alpha}\pounds_k \gamma_{ab} + \frac{1}{2\alpha}\pounds_\omega \gamma_{ab} = \frac{1}{2\alpha}\pounds_\omega \gamma_{ab}. \tag{317}$$

Writing spatial tracefree part of the Einstein equation,

$$(G_{\alpha\beta} - 8\pi T_{\alpha\beta})(\gamma_a{}^\alpha \gamma_b{}^\beta - \frac{1}{3}\gamma_{ab}\gamma^{\alpha\beta}) = \mathcal{E}_{ab} - \frac{1}{3}\tilde{\gamma}_{ab}\tilde{\gamma}^{cd}\mathcal{E}_{cd} = 0, \tag{318}$$

$$\begin{aligned} \mathcal{E}_{ab} &:= -\pounds_n K_{ab} + {}^3R_{ab} + KK_{ab} - 2K_{ac}K_b{}^c - \frac{1}{\alpha}D_aD_b\alpha - 8\pi S_{ab} \\ &= -\frac{1}{\alpha}\pounds_k K_{ab} + \frac{1}{\alpha}\pounds_\omega K_{ab} + {}^3\tilde{R}_{ab} + {}^3\tilde{R}^\psi_{ab} - \frac{1}{\alpha}D_aD_b\alpha + KK_{ab} - 2K_{ac}K_b{}^c - 8\pi S_{ab}, \end{aligned} \tag{319}$$

the trace free part of the spatial projection of $\pounds_k K_{ab}$ is written in the maximal slice,

$$\begin{aligned} &(\gamma_a{}^c\gamma_b{}^d - \frac{1}{3}\gamma_{ab}\gamma^{cd})\pounds_k K_{cd} \\ &= (\tilde{\gamma}_a{}^c\tilde{\gamma}_b{}^d - \frac{1}{3}\tilde{\gamma}_{ab}\tilde{\gamma}^{cd})\left[\pounds_\omega(\psi^4\tilde{A}_{cd}) + \alpha\ {}^3\tilde{R}_{cd} + \alpha\left({}^3\tilde{R}^\psi_{cd} - \frac{1}{\alpha}D_cD_d\alpha\right) - 2\alpha\psi^4\tilde{A}_{ce}\tilde{A}_d{}^e - 8\pi\alpha S_{cd}\right]. \end{aligned} \tag{320}$$

Note that the tracefree part of the extrinsic curvature A_{ab} is not that of the waveless (or IWM) K_{ab}, but the one defined by Eq. (317).

The spatial trace part of the Einstein equation is written,

$$(G_{\alpha\beta} - 8\pi T_{\alpha\beta})\gamma^{\alpha\beta} = 0 \tag{321}$$

$$\begin{aligned} (G_{\alpha\beta} - 8\pi T_{\alpha\beta})\gamma^{\alpha\beta} &= \frac{2}{\alpha}\gamma^{ab}\pounds_k K_{ab} - \frac{2}{\alpha}\pounds_\omega K + 4K^{ab}\left(K_{ab} - \frac{1}{2\alpha}\pounds_\omega\gamma_{ab}\right) \\ &+ \left(-\frac{1}{2}{}^3R + \frac{2}{\alpha}D^aD_a\alpha\right) - \frac{1}{2}(K^2 + 3K_{ab}K^{ab}) - 8\pi S = 0. \end{aligned} \tag{322}$$

Using the helical K_{ab} Eq. (317) again, $\gamma^{ab}\pounds_k K_{ab}$ is written in $K = 0$ gauge,

$$\gamma^{ab}\pounds_k K_{ab} = -\frac{\alpha}{2}\left(-\frac{1}{2}{}^3R + \frac{2}{\alpha}D^aD_a\alpha\right) + \frac{3}{4}\,\alpha\,\tilde{A}_{ab}\tilde{A}^{ab} + 4\pi\alpha S. \tag{323}$$

Then, we introduce integrals of $\pounds_k K_{ab}$ over the near zone to estimate the violation of the helical symmetry as follows;

$$||\pounds_k K_{ab}||_1 := \int_{r\le r_0}\sqrt{\gamma^{ac}\gamma^{bd}\pounds_k K_{ab}\pounds_k K_{cd}}\sqrt{\gamma}d^3x, \tag{324}$$

$$||\pounds_k K_{ab}||_2 := \sqrt{\int_{r\le r_0}\gamma^{ac}\gamma^{bd}\pounds_k K_{ab}\pounds_k K_{cd}\sqrt{\gamma}d^3x}, \tag{325}$$

where,

$$\pounds_k K_{ab} = (\gamma_a{}^c\gamma_b{}^d - \frac{1}{3}\gamma_{ab}\gamma^{cd})\pounds_k K_{cd} + \frac{1}{3}\gamma_{ab}\gamma^{cd}\pounds_k K_{cd}. \tag{326}$$

For the above norms, the integarls are unbounded if one extend the integral to infinity, $r_0 \to \infty$. For the norm,

$$||\pounds_k K_{ab}||_4 := \left[\int_{r\le r_0} \left(\gamma^{ac}\gamma^{bd}\pounds_k K_{ab}\pounds_k K_{cd}\right)^2 \sqrt{\gamma}d^3x\right]^{1/4}, \tag{327}$$

the integral is bounded for $r_0 \to \infty$. The waveless solution $\{\alpha, \beta^a, \psi, \tilde\gamma_{ab}\}^{\rm WL}$ or the IWM solution $\{\alpha, \beta^a, \psi\}^{\rm IWM}$ are substituted to the above $\{\alpha, \beta^a, \psi, \tilde\gamma_{ab}\}$, which is derived from the helically symmetric Einstein equations. The domain for integration is taken from the center of orbit to the certain radius r_0 for which we take $r_0/(\pi/\Omega) = 0.1, 1, 10$. Also compute the value over the fluid support and whole region.

N.B. 0.: These norms are normalized by the total gravitational mass of isolated binary neutron stars M_∞ as,

$$||\pounds_k K_{ab}||_1/M_\infty, \quad ||\pounds_k K_{ab}||_2\, M_\infty^{1/2}, \quad \text{and} \quad ||\pounds_k K_{ab}||_4\, M_\infty^{5/4}. \tag{328}$$

N.B. 1.: Definitions for the extrinsic curvature of the helically symmetric spacetime and the waveless formulation (+ helical symmetry) differs as follows:

$$K_{ab} = \frac{1}{2\alpha}\pounds_\omega \gamma_{ab}, \qquad \text{helical}, \tag{329}$$

$$K_{ab} = \frac{1}{2\alpha}\pounds_\beta \gamma_{ab} + \frac{1}{6\alpha}\gamma_{ab}\gamma^{cd}\pounds_{\Omega\phi}\gamma_{cd}, \qquad \text{waveless}. \tag{330}$$

However, the trace K are the same for both the helically symmetric spacetime and the waveless formulation,

$$K = \frac{1}{2\alpha}\gamma^{ab}\pounds_\omega \gamma_{ab} = 0. \tag{331}$$

Therefore the maximal slice condition $K = 0$ takes the same form for both cases.

N.B. 2.: One can analytically subtract the waveless equation from the Eqs.(320) and (323) to calculate the violation of helical symmetry,

$$(\gamma_a{}^c\gamma_b{}^d - \frac{1}{3}\gamma_{ab}\gamma^{cd})\pounds_k K_{cd} = (\tilde\gamma_a{}^c\tilde\gamma_b{}^d - \frac{1}{3}\tilde\gamma_{ab}\tilde\gamma^{cd})\left[\pounds_\omega(\psi^4\bar A_{cd}) - 2\alpha\psi^4\bar A_{ce}\bar A_d{}^e,\right] \tag{332}$$

$$\gamma^{ab}\pounds_k K_{ab} = \frac{3}{4}\,\alpha\,\bar A_{ab}\bar A^{ab}, \tag{333}$$

where

$$\bar A_{ab} := (\gamma_a{}^c\gamma_b{}^d - \frac{1}{3}\gamma_{ab}\gamma^{cd})\frac{1}{2\alpha}\left(\pounds_\omega\tilde\gamma_{cd} - \pounds_\beta\tilde\gamma_{cd}\right) = \frac{1}{2\alpha}\Omega(\tilde L\phi)_{ab}. \tag{334}$$

N.B. 3.: In Fig.1 and 2, the normalized norm $||\pounds_k K_{ab}||_2\, {M_\infty}^{1/2}$ is plotted against ΩM_∞. In Fig.1, two curves are drawn for the waveless sequence with $(M/R)_\infty = 0.17$. For the solid curve labeled "WL all terms", Eqs. (320) and (323), are used for computing the norm. For the dashed curve, labeled "WL removed", Eqs. (332) and (333) are used. Two curves agree at the larger ΩM_∞ (smaller separation), However, the result using Eqs. (320) and (323) seems erroneous at the smaller ΩM_∞ (larger separation). The reason for the disagreement hasn't been investigated yet, but it would be due to the resolution of the coordinate. (I mean a coordinate to compute the integral and the gravitational field, whose origine is centered at the orbit, and I computed the sequence with middle size resolution $N_r = 24$ because of the limitation of computing resources.)

It may be possible to roughly estimate the magnitude of the norm as follows. Assuming the Keplar law for the orbital motion, $(\Omega M)^2 = (M/2d)^3$ and $v^2 = (\Omega M)^{2/3}$, where M is the total mass, d, half of the separation (orbital radius), and $v = d\Omega$, the difference between an exactly helical and a waveless solution in $\pounds_k K_{ab}$ may be estimated as $\pounds_k K_{ab} \sim \partial_t^2\tilde\gamma_{ab} \sim \Omega^2 h$. ???? OR $\pounds_k K_{ab} \sim \partial_t^2\tilde\gamma_{ab} \sim \Omega^2(d/R)^2 h \sim v^2 h/R^2$. Further assume the magnitude of h_{ab} to be the second order post-Newtonian order, $h_{..} \sim (M/R)(\Omega M)^{2/3}$, the magnitude of normalized norm may be written

$$||\pounds_k K_{ab}||_2\, M^{1/2} \sim \left[(\Omega^2 h)^2 R^3\right]^{1/2} M^{1/2} \sim (M/R)^{-1/2}\,(\Omega M)^{8/3}, \tag{335}$$

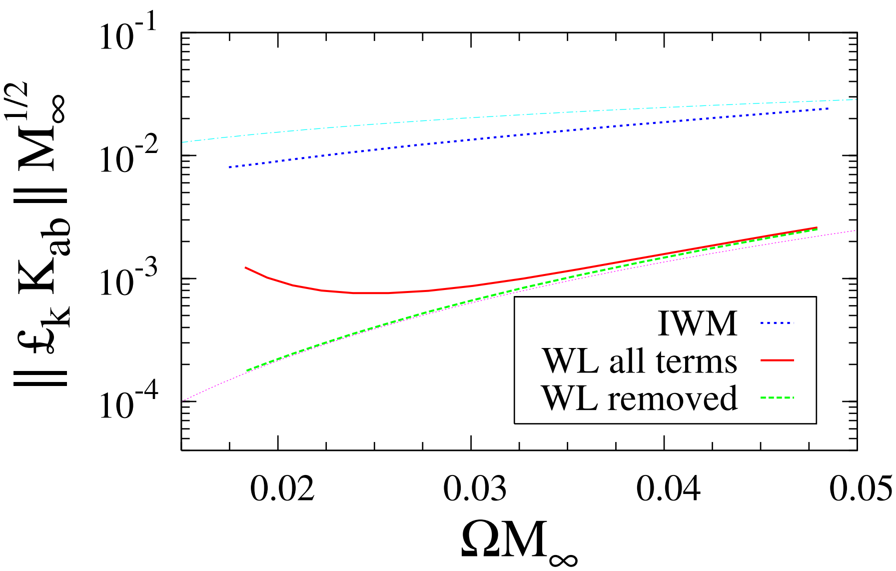

FIG. 1: For detail, see N.B. 3.

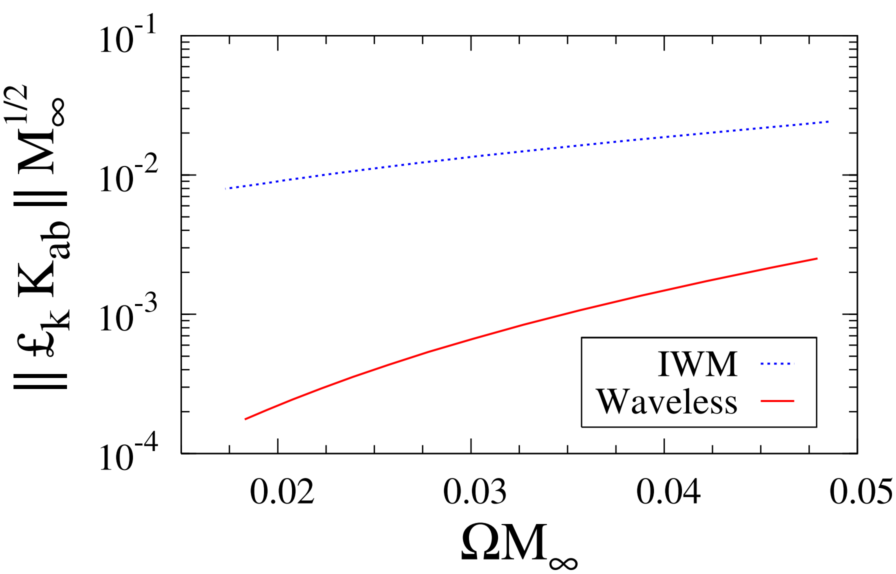

FIG. 2: For detail, see N.B. 3.

$$||\pounds_k K_{ab}||_2\, M^{1/2} \sim \left[(v^2 h/R^2)^2 R^3\right]^{1/2} M^{1/2} \sim (M/R)^{3/2}\,(\Omega M)^{4/3},$$

where R is the size of a neutron star. For the IWM sequence, the magnitude of $\pounds_k K_{ab}$ may be estimated from the fluid source term as $\pounds_k K_{ab} \sim \mathring{\Delta} h_{ab}\,(\sim S \sim \rho v^2)$, hence

$$||\pounds_k K_{ab}||_2\, M^{1/2} \sim \left[(h/R^2)^2 R^3\right]^{1/2} M^{1/2} \sim (M/R)^{3/2}\,(\Omega M)^{2/3}. \tag{336}$$

Factors are ignored in the above estimations. In Fig.1, the estimation for the IWM curve with a factor 3, i.e. $3(M/R)^{3/2}\,(\Omega M)^{2/3}$ (curve at the top in the figure), and for the waveless with a factor 3, i.e. $3(M/R)^{-1/2}\,(\Omega M)^{8/3}$ (curve at the bottom in the figure), are plotted together.

If everybody is agreed, I would like to replace the Fig.5 in the submitted paper with Fig. 2 presented above, in which the curve "WL removed" is plotted with the IWM curve.

VIII. A NOTE FOR THE GAUGE CONDITION

The harmonic coordinate condition to the spacetime manifold $\mathcal{M}$ is written

$$\nabla_\beta \pounds_t(\sqrt{-g}\, g^{\alpha\beta}) = 0. \tag{337}$$

A relation $\delta g = g g^{\alpha\beta}\delta g_{\alpha\beta}$ implies

$$\frac{1}{\sqrt{-g}}\pounds_t \sqrt{-g} \;=\; \nabla_\alpha t^\alpha, \tag{338}$$

$$\pounds_t \sqrt{-g} \;=\; \nabla_\alpha(\sqrt{-g}\, t^\alpha) \;=\; \overset{\circ}{\nabla}_\alpha(\sqrt{-g}\, t^\alpha), \tag{339}$$

$$\pounds_t(\sqrt{-g}\, g^{\alpha\beta}) \;=\; \sqrt{-g}\, \nabla_\gamma(\, t^\gamma g^{\alpha\beta}) \;-\; \sqrt{-g}(\, \nabla^\alpha\, t^\beta \;+\; \nabla^\beta\, t^\alpha). \tag{340}$$

Therefore,

$$\nabla_\beta \pounds_t(\sqrt{-g}\, g^{\alpha\beta}) \;=\; \sqrt{-g}(\nabla^\alpha\nabla_\beta t^\beta - \nabla_\beta\nabla^\alpha t^\beta - \nabla_\beta\nabla^\beta t^\alpha) \;=\; -\sqrt{-g}(R^\alpha{}_\beta t^\beta + \nabla_\beta\nabla^\beta t^\alpha) = 0, \tag{341}$$

or

$$\nabla^\beta\nabla_\beta t^\alpha \;+\; R^\alpha{}_\beta t^\beta \;=\; 0. \tag{342}$$

The radiation gauge is to impose the condition on the three manifold Σ,

$$D_b\partial_t(\gamma^{1/3}\gamma^{ab}) \;=\; 0 \tag{343}$$

WRONG:machigai
If the helical symmetry is satisfied $\partial_t(\sqrt{\gamma}\gamma^{ab}) = -\pounds_{\Omega\phi}(\sqrt{\gamma}\gamma^{ab})$, we have

$$D^b D_b(\Omega\phi^a) \;+\; R^a{}_b\Omega\phi^b \;=\; 0, \tag{344}$$

$$D^b D_b\phi^a \;+\; R^a{}_b\phi^b \;=\; 0. \tag{345}$$

($\Omega =$ constant does not lose the generality, since Ω is an angular velocity of the rotation of the frame, not the fluid.)
The Dirac gauge restricts this term in the following way.

APPENDIX A: 3+1 FORMULATION

We consider globally hyperbolic spacetimes $\mathcal{M}, g_{\alpha\beta}$ with perfect fluid sources. Our formulation is applied to spacetimes written in 3+1 form, with metric γ_{ab} on Cauchy surfaces Σ and its conjugate momentum π^{ab} as independent variables. Let t^α be a vector field transverse to a Cauchy surface $\Sigma = \Sigma_0$ and let $\Sigma_t = \chi_t(\Sigma_0)$ the family of slices with the diffeo χ_t generated by t^α. Let n^α be the future-pointing unit normal to this foliation. The projection 4-tensor $\gamma_{\alpha\beta}$ orthogonal to n_α, is defined as

$$\gamma_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta, \tag{A1}$$

which is associated with three metric $\gamma_{ab}(t)$ on the spatial slices Σ_t. Any spatial tensors on Σ_t can be identified with spacetime tensors which are orthogonal to n^α on all of their indices.

Although t^α may not necessary be everywhere timelike, it tends to be timelike at spatial infinity and is transverse to a family of spacelike hypersurfaces Σ_t, which allows to introduce a nonvanishing lapse α and a shift β^α that relate $\partial_t \equiv t^\alpha$ to n^α in the usual way,

$$t^\alpha = \alpha n^\alpha + \beta^\alpha, \qquad \beta^\alpha n_\alpha = 0. \tag{A2}$$

Then, in a chart $\{t, x^i\}$ for which Σ_t is a t =constant surface, the metric $g_{\alpha\beta} = \gamma_{\alpha\beta} - n_\alpha n_\beta$ has the form

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \tag{A3}$$

With D_a the covariant derivative associated with the spatial metric γ_{ab}, the extrinsic curvature of Σ_t is given by

$$K_{ab} = -\frac{1}{2}\pounds_n \gamma_{ab} = \frac{1}{2\alpha}(-\partial_t\gamma_{ab} + D_a\beta_b + D_b\beta_a), \tag{A4}$$

where γ_{ab}, $\partial_t\gamma_{ab}$, and $\pounds_n\gamma_{ab}$ are the pullback to Σ_t of $\gamma_{\alpha\beta}$, $\pounds_t\gamma_{\alpha\beta}$, and $\pounds_n\gamma_{\alpha\beta}$, respectively.

A perfect fluid is described by a stress tensor

$$T^{\alpha\beta} = (\epsilon + p)u^\alpha u^\beta + p g^{\alpha\beta}, \tag{A5}$$

where u^α, p and ϵ are the fluid four-velocity, pressure p and energy density, respectively. The pressure and the energy density are assumed to satisfy an equation of state of the form

$$p = p(\rho, s), \quad \epsilon = \epsilon(\rho, s) \tag{A6}$$

where ρ the baryon-mass density and s the entropy per unit baryon mass.

The Lagrangian density,

$$\mathcal{L} = \left(\frac{1}{16\pi}R - \epsilon\right)\sqrt{-g}, \tag{A7}$$

is written in terms of ADM variables, the spatial metric γ_{ab} and its conjugate momentum

$$\pi^{ab} = -(K^{ab} - \gamma^{ab}K)\sqrt{\gamma} \tag{A8}$$

as

$$\begin{aligned} 16\pi\mathcal{L} &= \pi^{ab}\partial_t\gamma_{ab} - \alpha\mathcal{H}_{\rm G} - \beta_a\mathcal{C}^a_{\rm G} + D_a(-2D^a\alpha\sqrt{\gamma} - 2\beta^b\pi^a{}_b + \beta^a\pi) - \partial_t\pi - 16\pi\,\epsilon\sqrt{-g} &\text{(A9)}\\ &= \pi^{ab}\partial_t\gamma_{ab} - \alpha\mathcal{H} - \beta_a\mathcal{C}^a + D_a(-2D^a\alpha\sqrt{\gamma} - 2\beta^b\pi^a{}_b + \beta^a\pi) - \partial_t\pi + 16\pi\, j_a v^a\sqrt{\gamma}\,, &\text{(A10)}\end{aligned}$$

where we have set

$$\mathcal{H}_{\rm G} := -2G^{\alpha\beta}n_\alpha n_\beta\sqrt{\gamma}\,, \qquad \mathcal{C}^a_{\rm G} := -2G^{\alpha\beta}\gamma_\alpha{}^a n_\beta\sqrt{\gamma} \tag{A11}$$

$$\rho_H := T^{\alpha\beta}n_\alpha n_\beta, \quad j^a := -T^{\alpha\beta}\gamma_\alpha{}^a n_\beta, \tag{A12}$$

$$\begin{aligned} \mathcal{H} &:= -2(G^{\alpha\beta} - 8\pi T^{\alpha\beta})n_\alpha n_\beta\sqrt{\gamma} = -\left[{}^3R - \frac{1}{\gamma}\left(\pi_{ab}\pi^{ab} - \frac{1}{2}\pi^2\right) - 16\pi\rho_H\right]\sqrt{\gamma}, &\text{(A13)}\\ \mathcal{C}^a &:= -2(G^{\alpha\beta} - 8\pi T^{\alpha\beta})\gamma_\alpha{}^a n_\beta\sqrt{\gamma} = -2(D_b\pi^{ab} + 8\pi j^a\sqrt{\gamma}). &\text{(A14)}\end{aligned}$$

Here, v^α is introduced by the following decomposition of the four velocity $u^\alpha = u^t(t^\alpha + v^\alpha)$ with $v^\alpha n_\alpha = 0$.

For a convenience of calculating the variation of the action, we follow Routh procedure, that is, a family of perfect-fluid spacetimes is specified by the canonical variables for the gravitational field, while the Lagrangian variables for the fluid as

$$\mathcal{Q}(\lambda) := [\gamma_{ab}(\lambda), \pi^{ab}(\lambda), \alpha(\lambda), \beta^a(\lambda), u^\alpha(\lambda), \rho(\lambda), s(\lambda)]. \tag{A15}$$

The Eulerian change in each quantity is defined by $\delta\mathcal{Q} := \frac{d}{d\lambda}\mathcal{Q}(\lambda)$. We introduce a Lagrangian displacement ξ^α in the following way: Let $\mathcal{Q} := \mathcal{Q}(\lambda)$, and let Ψ_λ be a diffeo mapping each trajectory (worldline) of the initial fluid to a corresponding trajectory of the configuration $\mathcal{Q}(\lambda)$. Then the tangent $\xi^\alpha(P)$ to the path $\lambda \to \Psi_\lambda(P)$ can be regarded as a vector joining the fluid element at P in the configuration $\mathcal{Q}(\lambda)$ to a fluid element in a nearby configuration. The Lagrangian change in a quantity at $\lambda = 0$ is then given by

$$\Delta\mathcal{Q} := \frac{d}{d\lambda}\Psi_{-\lambda}\mathcal{Q}(\lambda)|_{\lambda=0} = (\delta + \pounds_\xi)\mathcal{Q}. \tag{A16}$$

The description of fluid perturbation in terms of a Lagrangian displacement ξ^α has a gauge freedom since a class of trivial displacement does not yield any Eulerian change in the fluid variables. We use this freedom to choose a gauge in which $\xi^t := \xi^\alpha\nabla_\alpha t = 0$, following [?].

The variation of the Lagrangian density (A10) is computed as

$$\begin{aligned}\delta\mathcal{L} \ =\ & \frac{1}{16\pi}\left\{ -\delta\alpha\mathcal{H} \ -\ \delta\beta^a\mathcal{C}_a \ +\ \delta\pi^{ab}\left[\partial_t\gamma_{ab} - D_a\beta_b - D_b\beta_a - \frac{2\alpha}{\sqrt{\gamma}}\left(\pi_{ab} - \frac{1}{2}\gamma_{ab}\pi\right)\right] \ -\ \delta\gamma_{ab}(G^{ab} - 8\pi S^{ab})\alpha\sqrt{\gamma}\right\} \\ & -\rho T\sqrt{-g}\Delta s \ -\ \frac{h}{u^t}\Delta(\rho u^t\sqrt{-g}) \ -\ \xi^\alpha\nabla_\beta T_\alpha{}^\beta\sqrt{-g} \ +\ D_a\tilde{\Theta}^a\sqrt{\gamma} \ -\ \frac{1}{16\pi}\partial_t(\delta\pi^{ab}\gamma_{ab}) \ +\ \partial_t(j_a\xi^a\sqrt{\gamma})\ ,\end{aligned} \tag{A17}$$

where G^{ab} and S^{ab} are the spatial projection of Einstein tensor and stress tensor, respectively, defined as

$$G^{ab} := G^{\alpha\beta}\gamma_\alpha{}^a\gamma_\beta{}^b, \quad \text{and} \quad S^{ab} := T^{\alpha\beta}\gamma_\alpha{}^a\gamma_\beta{}^b, \tag{A18}$$

$$\begin{aligned}G^{ab} = \ & \frac{1}{\alpha\sqrt{\gamma}}(\partial_t\pi^{ab} - \pounds_\beta\pi^{ab}) \ +\ {}^3R^{ab} - \gamma^{ab}\,{}^3R + \frac{2}{\gamma}\left(\pi^{ac}\pi^b{}_c - \frac{1}{2}\pi\pi^{ab}\right) - \frac{1}{\alpha}(D^aD^b\alpha - \gamma^{ab}D^cD_c\alpha) \\ & +\ \frac{1}{2}\gamma^{ab}\left[{}^3R - \frac{1}{\gamma}\left(\pi^{cd}\pi_{cd} - \frac{1}{2}\pi^2\right)\right].\end{aligned} \tag{A19}$$

The term relating to the divergence of the stress tensor can be written in the spatial quantities as

$$\xi^\alpha\nabla_\beta T_\alpha{}^\beta\sqrt{-g} = \xi^a\left[\,\partial_t(j_a\sqrt{\gamma}) - \pounds_\beta(j_a\sqrt{\gamma}) + D_b(S_a{}^b\alpha\sqrt{\gamma}) + \rho_H D_a\alpha\sqrt{\gamma}\,\right]. \tag{A20}$$

$\tilde{\Theta}^a$ is the surface term of the Lagrangian density,

$$\begin{aligned}\tilde{\Theta}^a \ =\ & \frac{1}{16\pi}\left\{[-2\delta(D^a\alpha\sqrt{\gamma}) + \beta^a\gamma_{bc}\delta\pi^{bc} + \pi\delta\beta^a - 2\pi^a{}_b\delta\beta^b]\frac{1}{\sqrt{\gamma}}\right. \\ & \left. +\ (\gamma^{ac}\gamma^{bd} - \gamma^{ab}\gamma^{cd})(\alpha D_b\delta\gamma_{cd} - D_b\alpha\,\delta\gamma_{cd}) + \alpha(\epsilon + p)q^a{}_b\xi^b - \beta^a j_b\xi^b\ \right\},\end{aligned} \tag{A21}$$

where $q^{\alpha\beta} := g^{\alpha\beta} + u^\alpha u^\beta$ is the projection tensor orthogonal to u^α and q^{ab} is its projection onto Σ_t, $q^{ab} := q^{\alpha\beta}\gamma_\alpha{}^a\gamma_\beta{}^b$.

Independent variations for field variables $\{\alpha, \beta^a, \pi^{ab}, \gamma_{ab}\}$ results in the field equations in ADM formalism,

$$\mathcal{H} = 0, \quad \mathcal{C}_a = 0, \quad \text{and} \quad G^{ab} = 8\pi S^{ab}, \tag{A22}$$

with a relation

$$\frac{2\alpha}{\sqrt{\gamma}}\left(\pi_{ab} - \frac{1}{2}\gamma_{ab}\pi\right) = \partial_t\gamma_{ab} - D_a\beta_b - D_b\beta_a \tag{A23}$$

APPENDIX B: MOMENTUM CONSTRAINTS IN TERMS OF ROTATING SHIFT

a. full form

$$(G_{\alpha\beta} - 8\pi T_{\alpha\beta})\gamma_a{}^\alpha n^\beta = 0 \tag{B1}$$

$$\begin{aligned}(G_{\alpha\beta} - 8\pi T_{\alpha\beta})\gamma_a{}^\alpha n^\beta &= -D_b K_a{}^b + D_a K + 8\pi j_a &(B2)\\ &= -\frac{1}{\sqrt{\gamma}} D_b\left(\sqrt{\gamma} A_a{}^b\right) + \frac{2}{3} D_a K + 8\pi j_a &(B3)\end{aligned}$$

b. Reduction

With $K=0, \pounds_k g_{\alpha\beta}=0$

$$-\frac{1}{\sqrt{\gamma}} D_b\left(\sqrt{\gamma} A_a^b\right) + \frac{2}{3} D_a K + 8\pi j_a = -\frac{1}{\sqrt{\gamma}}\tilde{D}_b\left(\sqrt{\gamma}\tilde{A}_a{}^b\right) + 8\pi j_a = -\frac{1}{\psi^6}\tilde{D}_b\left(\psi^6 \tilde{A}_a{}^b\right) + 8\pi j_a = 0 \tag{B4}$$

Substitute

$$\tilde{A}_a{}^b = \frac{1}{2\alpha}\left(\tilde{D}_a\tilde{\omega}^b + \tilde{D}^b\tilde{\omega}_a - \frac{2}{3}\tilde{\gamma}_a{}^b \tilde{D}_c\omega^c\right) \tag{B5}$$

$$\begin{aligned}&\frac{1}{\psi^6}\tilde{D}_b\left[\psi^6\frac{1}{2\alpha}\left(\tilde{D}_a\tilde{\omega}^b + \tilde{D}^b\tilde{\omega}_a - \frac{2}{3}\tilde{\gamma}_a{}^b\tilde{D}_c\omega^c\right)\right] = 8\pi j_a &(B6)\\ \Leftrightarrow\quad &\frac{1}{2}\tilde{D}_b\left(\tilde{D}_a\tilde{\omega}^b + \tilde{D}^b\tilde{\omega}_a - \frac{2}{3}\tilde{\gamma}_a{}^b\tilde{D}_c\omega^c\right) + \tilde{A}_a{}^b\frac{\alpha}{\psi^6}\tilde{D}_b\left(\frac{\psi^6}{\alpha}\right) = 8\pi j_a &(B7)\\ \Leftrightarrow\quad &\tilde{D}_b\tilde{D}_a\tilde{\omega}^b + \tilde{D}_b\tilde{D}^b\tilde{\omega}_a - \frac{2}{3}\tilde{D}_a\tilde{D}_c\omega^c + 2\alpha\tilde{A}_a{}^b\frac{\alpha}{\psi^6}\tilde{D}_b\left(\frac{\psi^6}{\alpha}\right) = 16\pi\alpha j_a &(B8)\end{aligned}$$

Using

$$\tilde{D}_b\tilde{D}_a\omega^b - \tilde{D}_a\tilde{D}_b\omega^b = {}^3\tilde{R}_{ab}\omega^b \tag{B9}$$

$$\tilde{D}_b\tilde{D}^b\tilde{\omega}_a + \frac{1}{3}\tilde{D}_a\tilde{D}_b\tilde{\omega}^b + {}^3\tilde{R}_{ab}\tilde{\omega}^b + 2\alpha\tilde{A}_a{}^b\frac{\alpha}{\psi^6}\tilde{D}_b\left(\frac{\psi^6}{\alpha}\right) = 16\pi\alpha j_a \tag{B10}$$

c. Shibata bunkai

Shibata decomposition: from $\tilde{D}_b\tilde{D}^b\tilde{\omega}_a + \frac{1}{3}\tilde{D}_a\tilde{D}_b\tilde{\omega}^b$ separate $\mathring{D}_b\mathring{D}^b\tilde{\omega}_a + \frac{1}{3}\mathring{D}_a\mathring{D}^b\tilde{\omega}_b$ then apply

$$\tilde{\omega}_a = G_a + \frac{1}{8}\mathring{D}_a(B - x^b G_b), \quad \text{where} \quad \mathring{D}_a x^b = \delta_a^b \tag{B11}$$

$$\begin{aligned}\tilde{D}_b\tilde{D}^b\tilde{\omega}_a = \tilde{\gamma}^{bc}\tilde{D}_b\tilde{D}_c\tilde{\omega}_a &= \tilde{\gamma}^{bc}\left[\mathring{D}_b(\tilde{D}_c\tilde{\omega}_a) - C^d_{bc}\tilde{D}_d\tilde{\omega}_a - C^d_{ba}\tilde{D}_c\tilde{\omega}_d\right] &(B12)\\ &= \tilde{\gamma}^{bc}\left[\mathring{D}_b(\mathring{D}_c\tilde{\omega}_a - C^d_{ca}\tilde{\omega}_d) - C^d_{bc}\tilde{D}_d\tilde{\omega}_a - C^d_{ba}\tilde{D}_c\tilde{\omega}_d\right] &(B13)\\ &= \mathring{\Delta}\tilde{\omega}_a + h^{bc}\mathring{D}_b\mathring{D}_c\tilde{\omega}_a - \tilde{\gamma}^{bc}\mathring{D}_b(C^d_{ca}\tilde{\omega}_d) - \tilde{\gamma}^{bc}C^d_{bc}\tilde{D}_d\tilde{\omega}_a - \tilde{\gamma}^{bc}C^d_{ba}\tilde{D}_c\tilde{\omega}_d &(B14)\end{aligned}$$

$$\tilde{D}_a\tilde{D}_b\tilde{\omega}^b = \mathring{D}_a(\tilde{\gamma}^{bc}\tilde{D}_b\tilde{\omega}_c) = \mathring{D}_a\left[f^{bc}(\mathring{D}_b\tilde{\omega}_c - C^d_{bc}\tilde{\omega}_d) + h^{bc}\tilde{D}_b\tilde{\omega}_c\right] = \mathring{D}_a\mathring{D}^b\tilde{\omega}_b + \mathring{D}_a(-f^{bc}C^d_{bc}\tilde{\omega}_d + h^{bc}\tilde{D}_b\tilde{\omega}_c) \tag{B15}$$

$$= \mathring{D}_a\left[\tilde{\gamma}^{bc}(\mathring{D}_b\tilde{\omega}_c - C^d_{bc}\tilde{\omega}_d)\right] = \mathring{D}_a\mathring{D}^b\tilde{\omega}_b + \mathring{D}_a(h^{bc}\mathring{D}_b\tilde{\omega}_c - \tilde{\gamma}^{bc}C^d_{bc}\tilde{\omega}_d) \tag{B16}$$

$$\begin{aligned}\tilde{D}_b\tilde{D}^b\tilde{\omega}_a + \frac{1}{3}\tilde{D}_a\tilde{D}_b\tilde{\omega}^b = \quad & \mathring{\Delta}\tilde{\omega}_a + \frac{1}{3}\mathring{D}_a\mathring{D}^b\tilde{\omega}_b + h^{bc}\mathring{D}_b\mathring{D}_c\tilde{\omega}_a - \tilde{\gamma}^{bc}\mathring{D}_b(C^d_{ca}\tilde{\omega}_d) \\ & -\tilde{\gamma}^{bc}C^d_{bc}\tilde{D}_d\tilde{\omega}_a - \tilde{\gamma}^{bc}C^d_{ba}\tilde{D}_c\tilde{\omega}_d + \frac{1}{3}\mathring{D}_a(-f^{bc}C^d_{bc}\tilde{\omega}_d + h^{bc}\tilde{D}_b\tilde{\omega}_c)\end{aligned} \tag{B17}$$

$$\begin{aligned}= \quad & \mathring{\Delta}\tilde{\omega}_a + \frac{1}{3}\mathring{D}_a\mathring{D}^b\tilde{\omega}_b + h^{bc}\mathring{D}_b\mathring{D}_c\tilde{\omega}_a - \tilde{\gamma}^{bc}\mathring{D}_b(C^d_{ca}\tilde{\omega}_d) \\ & -\tilde{\gamma}^{bc}C^d_{bc}\tilde{D}_d\tilde{\omega}_a - \tilde{\gamma}^{bc}C^d_{ba}\tilde{D}_c\tilde{\omega}_d + \frac{1}{3}\mathring{D}_a(h^{bc}\mathring{D}_b\tilde{\omega}_c - \tilde{\gamma}^{bc}C^d_{bc}\tilde{\omega}_d)\end{aligned} \tag{B18}$$

Substitute $\tilde{\omega}_a = G_a + \frac{1}{8}\mathring{D}_a(B - x^bG_b)$,

$$\mathring{\Delta}\tilde{\omega}_a + \frac{1}{3}\mathring{D}_a\mathring{D}^b\tilde{\omega}_b = \mathring{\Delta}G_a + \frac{1}{8}\mathring{D}_a\mathring{\Delta}(B - x^bG_b) + \frac{1}{3}\mathring{D}_a\mathring{D}^b\left[G_b + \frac{1}{8}\mathring{D}_b(B - x^cG_c)\right] \tag{B19}$$

$$= \mathring{\Delta}G_a + \frac{1}{6}\mathring{D}_a(\mathring{\Delta}B - x^bG_b) \tag{B20}$$

Finally, solve separately

$$\begin{aligned}\mathring{\Delta}G_a = \quad & -h^{bc}\mathring{D}_b\mathring{D}_c\tilde{\omega}_a + \tilde{\gamma}^{bc}\mathring{D}_b(C^d_{ca}\tilde{\omega}_d) + \tilde{\gamma}^{bc}C^d_{bc}\tilde{D}_d\tilde{\omega}_a + \tilde{\gamma}^{bc}C^d_{ba}\tilde{D}_c\tilde{\omega}_d - \frac{1}{3}\mathring{D}_a(h^{bc}\mathring{D}_b\tilde{\omega}_c - \tilde{\gamma}^{bc}C^d_{bc}\tilde{\omega}_d) \\ & -{}^3\tilde{R}_{ab}\tilde{\omega}^b - 2\alpha\tilde{A}_a{}^b\frac{\alpha}{\psi^6}\tilde{D}_b\left(\frac{\psi^6}{\alpha}\right) + 16\pi\alpha j_a\end{aligned} \tag{B21}$$

and

$$\mathring{\Delta}B = x^bG_b \tag{B22}$$

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